SYMMETRY ANALYSIS OF DIFFERENTIAL EQUATIONS FOR COMPOSITES

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Introduction

Multi-scale phenomena which include modification of quantum states of materials caused by mechanical strains, ferroelectric transformations induced by electric field and mechanical stresses, chemical reaction processes biased by mechanical stresses and change of bio-molecular conformality of proteins caused by environmental mechanical strain rates, etc [1] often are modelled into differential equations which in most certain instances potent complicities and thus their solutions are often difficult to attain. Most times their solutions are numerically in the approximate forms. However Lie [2] symmetry groups provided classical means of overcoming the lacuna between the modelling of these scientific problems and their phenomena solutions. Since the work of Lie [2] which was primarily developed to generalize the method for solving differential equations in a holistic way the scene had not been the same as the result of his technique had found wide applications beyond the primary frontiers of his initial idea. This monumental art turns out to be fundamental phenomenon applicable to all spheres of pure and applied mathematics and engineering [1], [3], [4]. Today scientific collaborations requisitions demand the establishment of an engineering science base, a link between the discoveries of basic science and the design of commercial devices must be completed to realize the full potential of these collaborations [1]. This paper considered the application of Lie symmetry groups to two problems in composites engineering and obtain all their possible solutions. First we solve for all possible solutions of the governing equation of the Solid-Gas Reactions model, particularly the reaction with a solid particle (taking a spherical particle) and a reactant at the surface of the sphere, and the reaction takes place, consuming part of the solid and leaving behind a porous char. The reactants then diffuse through the porous char in order to react inner zone [5], [6]. The second problem considered is the problems with phase change, when liquid freezes or a solid melts, the phase change takes place at an interface that is at the freezing or melting point; this interface moves with time [6]. Problems of this type arise in metal casting, welding, and food processing, freeze-coating of fibbers', laser glazing, and the use of latent heat devices for energy conservation [6].

Problem Formulation and Procedures

The governing differential equation for Solid-Gas Reactions in considered is given by

$$C_r = \tfrac{1}{\gamma}[C_{rr} + \tfrac{2}{r}C_r] \qquad (1)$$

with boundary conditions $C(1,t) = 1$,

$$C(r_C, t) = 0\,,\ \frac{dr_C}{dt} = -\left.\frac{\partial C}{\partial r}\right|_{r=r_C}\ ;\ r_C(0) = 1 \qquad (2)$$

where C is the concentration of reactants; r_C is the radius of unreacted solid [5]. The governing differential equation for the problem with phase change is governed by

$$T_{i,t} = (\tfrac{k_i}{C_i\rho_i})T_{i,xx}\ ;\ i = 1,2 \qquad (3)$$

with temperature specified at two boundaries $T_1(0,t) = T_0$; $T_1(L,t) = T_L$ and initial condition taken as $T_i(x,0) = T_{init}(x)$ [5].

Lie symmetry groups' method of solutions

Let

$$\Gamma = \xi(t,r,C)\partial_r + \tau(t,r,C)\partial_t + \phi(t,r,C)\partial_C \quad (4)$$

be the infinitesimal Lie symmetry generators for the equation (1). Then the second prolongation of (4) is denoted by

$$\Gamma^{(2)} = \Gamma + \phi^r \partial_{C_r} + \phi^t \partial_{C_t} + \phi^{rr} \partial_{C_{rr}} + \phi^{rt} \partial_{C_{rt}} + \phi^{tt} \partial_{C_{tt}}, \quad (5)$$

the action of (5) on (1) is invariant that is

$$\Gamma^{(2)} \{C_r - \tfrac{1}{\gamma}[C_{rr} + \tfrac{2}{r} C_r]\} = 0. \quad (6)$$

For the case of (3) if we replace the respective variables C and r, in the infinitesimal and its second prolongation by T_i and x respectively and the action of the corresponding second prolongation on (3) will produce the invariant equation denoted by

$$\Gamma^{(2)} \{T_{i,t} - (\tfrac{k_i}{C_i \rho_i}) T_{i,xx}\} = 0. \quad (7)$$

Solving the symmetry equations (6) and (7) we obtain the following results respectively.

Results and Discussion

All possible solutions of (1) are as following:

$$C^{(1)} = \beta(r, t-\lambda),\; C^{(2)} = \beta(r-\lambda, t)\exp(\tfrac{\lambda}{r-\lambda}),$$

$$C^{(3)} = \beta(r-\lambda, t)\exp(C\lambda),\; C^{(5)} = \beta(re^{\lambda}, te^{2\lambda}),$$

$$C^{(4)} = \beta(re^{\lambda}, t)\exp[(\tfrac{2t}{r}e^{-\lambda} + \tfrac{r\gamma}{2}e^{\lambda})\lambda], \quad (8)$$

$$C^{(6)} = \beta(\tfrac{r}{1-4\lambda t}, \tfrac{t}{1-4\lambda t})\exp\{\tfrac{\lambda}{1-4\lambda t}(\tfrac{\gamma r^2}{1-4\lambda t} + 6t)\},$$

$$C^{(\alpha)} = \beta(r,t) + \lambda\alpha(r,t).$$

Where $\beta(r,t)$ is a solution satisfying the symmetry equation (6), $\alpha(r,t)$ is any other solution of the diffusion equation (1) and λ is a Lie group parameter. That is for c_1, c_3, c_4 and c_6 been arbitrary real constants we have that

$$\beta(r,t) = -c_1 r^{-1} - c_3(2r^{-1}t + \tfrac{r\gamma}{2}) - c_6(\gamma r^2 + 6t) + c_4. \quad (9)$$

All possible solutions of (3) are as following:

$$T_i^{(1)} = \beta(x, t-\lambda),\; T_i^{(2)} = \beta(x-\lambda, t),$$

$$T_i^{(3)} = \beta(x,t)\exp[(\tfrac{\rho_i c_i}{2k_i})\lambda],\; T_i^{(4)} = \beta(x,t)e^{-\lambda},$$

$$T_i^{(5)} = \beta(xe^{\lambda}, te^{2\lambda})\exp[-(\tfrac{\rho_i c_i}{k_i}te^{2\lambda})\lambda], \quad (10)$$

$$T_i^{(6)} = \beta(\tfrac{x}{1-4\lambda t}, \tfrac{t}{1-4\lambda t})\exp\{\tfrac{\lambda}{(1-4\lambda t)^2}(\tfrac{\rho_i c_i}{2k_i})(x^2 + t^2)\},$$

$$T_i^{(\alpha)} = \beta(x,t) + \lambda\alpha(x,t).$$

Where $\beta(x,t)$ is a solution satisfying the symmetry equation (7), $\alpha(x,t)$ is any other solution of the diffusion equation (3) and λ is a Lie group parameter.

That is for c_3, c_4, c_5 and c_6 been arbitrary real constants we have that

$$\beta(x,t) = -c_5(\tfrac{\rho_i c_i}{k_i})t - c_6\{(2\tfrac{\rho_i c_i}{k_i})[x^2 + t^2] - c_3(\tfrac{\rho_i c_i}{k_i}) + c_4. \quad (11)$$

Having obtained these solutions it is easy to deduce those solutions that satisfy the boundary conditions.

Conclusion

In the above the solution spaces of these two composite models are obtained via their Lie symmetry groups. It is easy to choose the optimum solutions from their solution spaces.

Reference

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