

# Linear Analytical Study of Large Deflection of Elastically-Bossed Sensor Plate due to Lateral Load

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## I. Abstract

The linear problem of large deflection of a pre-stressed layered plate with an elastic boss due to lateral pressure is studied. The approach extends von Karman plate theory for large deflection to a symmetrically layered plate including an elastic boss. The thus derived nonlinear governing equations are simplified by neglecting the arising nonlinear terms, yielding a modified Bessel equation for the lateral slope. Analytical solutions expressible in terms of modified Bessel functions are developed by considering the continuity at the interface between the boss and the annular plate as well as the boundary condition along the clamped edge. For a nearly monolithic plate with a thin boss, the solutions agree well with those of a single-layered plate available in literature. Various boss sizes and ratios between the Young's moduli of the boss and the plate are implemented with initial tension ranging from nearly zero to almost infinity. The radial variations of geometrical responses were thoroughly explored and the results show that, both the boss size and the modulus ratio can influence the geometrical behaviour of the bossed plate significantly.

## II. Introduction

Large deflection is often encountered for a micro sensor or actuator in practical application. A more advanced theory beyond Kirchhoff hypotheses of classical plate theory is required to predict the associated structural behavior. In addition, a sensing device may often be fabricated in a layered configuration with the presence of initial stresses. For instance, Wang et al. [1] conducted a mechanical analysis for a composite piezoelectric circular plate with initial tension. An analytical approach was developed by considering different material properties in each layer but only classical lamination theory was utilized [1]. In literature, the problem of initial tension has been well noted that it may degrade pressure sensitivity for a typical deflection-based sensing device [2]. The effect of pretension is thus of great importance if the structure undergoes a large deflection condition. In practical application, a bossed sensing device may have several advantages compared to a conventional one that is flat in geometry. This has been illustrated by Ettouhami, et al. [3] that it may increase both the sensitivity and linearity. The present study is thus motivated to study the problem of large deflection of an elastically-bossed layered plate under initial tension. von Karman's plate theory for

large deflection is employed to consider a symmetrically layered case including an elastic boss. To have an informative insight, however, only linear problem is considered. And thus the arising nonlinear terms in the governing equations will be neglected. The problem reduces to a modified Bessel equation for the lateral slope subjected to the continuity condition along the boss edge and the clamped boundary condition. Analytical solutions for the lateral slope were developed by imposing both the interface continuity at the boss edge and the boundary conditions at clamped end. The associated geometrical responses of the plate, such as the lateral deflection, lateral curvature, as well as those at the center of the plate were obtained subsequently. Various ratios in thickness, in-plane sizes and material properties between the boss and the annular plate were implemented with initial tension ranging from nearly zero to almost infinity. The effects due to the initial tension, as well as the boss size and modulus ratio between the boss and the annular plate upon the structural responses are thoroughly investigated.

## III. Formulation of the Problem and Approach

A clamped circular symmetrically layered plate with an elastic boss is considered. It is under a uniform in-plane tension,  $N_0$ , and a uniform lateral load,  $P_z=p_0$ , as shown in Fig. 1. Young's modulus, Poisson's ratio and thickness for the boss are  $E_b$ ,  $\nu_b$ , and  $h_b$ ; and those for a typical  $i^{th}$  layer in the annular plate are  $E_i$ ,  $\nu_i$ , and  $h_i$  respectively. Via von Karman's theory for large deflection, the governing equations for both the elastic boss and the annular layered plate can all be formulated in terms of radial slope and incremental in-plane force resultants, first. Following a non-dimensional scheme similar to Sheplak and Dugundji [4], these equations can further be simplified and merged to read,

$$\begin{cases} \xi^2 \theta'' + \xi \theta' - [1 + \xi^2 (k^2 + 12S_r)] \theta = 6P\xi^3 \\ \xi^2 S_r'' + 3\xi S_r' = -\frac{h^2}{2A_i D_i' \xi} \theta^2 \end{cases} \quad (1)$$

Where  $S_r$  is the non-dimensional incremental radial force resultants,  $P$  is the normalized lateral load;  $\xi$ ,  $\theta$ , and  $k$  are the dimensionless radial coordinate, lateral slope, and tension parameter, respectively. In additions,  $h$  is the thickness of the annular layered plate;  $A_i$  and  $D_i'$  are quantities defined by extensional and bending stiffness of the layered plate; and  $\theta' = d\theta/d\xi, \dots$  etc. The related non-dimensional quantities are defined as,

$$\xi = \frac{r}{r_a}, \quad W = \frac{w}{h}; \quad \theta = \frac{dW}{d\xi} = \frac{r_a}{h} \frac{dw}{dr}; \quad \Psi = \frac{d\theta}{d\xi} = \frac{r_a^2}{h} \frac{d^2w}{dr^2} = \dot{\theta};$$

$$[S_r, S_\theta] = [N_r, N_\theta] \frac{r_a^2}{D_l'}; \quad P = \frac{P_0 r_a}{h D_l'}$$

$$k = \sqrt{N_\theta r_a^2 / D_l'}; \quad (D_l', D_l'') = 12(D_l, D_l'')$$

Considering the case of small deflection, all the nonlinear terms in equations (1) can be neglected, yielding a modified Bessel equation for the lateral slope, i. e.,

$$\xi^2 \theta_{b(a)}'' + \xi \theta_{b(a)}' - (1 + k_{b(a)}^2 \xi^2) \theta_{b(a)} = 6P_{b(a)} \xi^3 \quad (2)$$

Where a subscript “b(a)” indicates that it applies for the boss (annular plate) region. The general solution for the lateral slope can therefore be given as,

$$\theta_{b(a)}(\xi) = C_{1(3)} I_1(k_{b(a)} \xi) + C_{2(4)} K_1(k_{b(a)} \xi) - \frac{6P_{b(a)} \xi}{k_{b(a)}^2} \quad (3)$$

In the above,  $I_1(x)$  and  $K_1(x)$  are the Modified Bessel functions of the 1<sup>st</sup> and the 2<sup>nd</sup> kind; and among the unknown constants,  $C_1, C_2$  can be found to be zero due to the asymptotic behavior of the function  $K_1(x)$  as  $x$  approaches to zero, for finite slope at the center of the elastic boss. The remaining unknown constants can be solved by considering the boundary conditions of the problem that (i) At the clamped edge :  $\theta_a(\xi=1)=0$ ; and (ii) Interface continuity (including lateral slope and moment resultant) at the boss edge :  $\theta_b(\xi_b) = \theta_a(\xi_b)$ ;  $M_{rb}(\xi_b) = M_{ra}(\xi_b)$ .

A set of 3 simultaneous equations for  $C_1, C_3$ , and  $C_4$  can thus be established. Upon solving for the simultaneous equations, the lateral slope in both the elastic boss and annular plate area can be obtained. A subsequent integration and differentiation can provide the solutions for lateral deflection and curvature, respectively.

#### IV. Numerical Remarks

For demonstration, only the lateral slopes for the cases of a nearly monolithic annular layered plate with  $E_1/E_2=1.05$ , and  $\nu_1=\nu_2=0.27$  [4] are presented. To have a check against the present approach, however, a thin boss with a small radius, i. e.,  $h_a/h_b=1.04$  and  $\xi_b=0.05$  simulating a nearly un-bossed (NUB) plate is considered, first. The lateral slopes for various initial tensions are presented in Fig. 2, including the results of Sheplak and Dugundji [4]. Apparently, a good agreement is obtained and thus the developed approach is validated. For the same annular plate, various boss sizes and ratios in thickness between the boss and the annular plate are further implemented. The results are given in Fig. 3. It can be seen, both the boss size and the thickness ratio can have a significant influence upon the behavior of lateral slope. Along both the boss and the clamped edges, a severe variation may exist even in the case of comparatively large initial tension and thus, the plate mode tends to prevail near these regions if a relatively large boss with apparent boss thickness is employed. On the other hand, the deformation of the elastic boss appears to be visual, particularly when a large radius for the boss considered with a comparable thickness to the annular layered plate.

#### V. Acknowledgements

The financial support provided for this study by the National Science Council in Taiwan through grant No. NSC 97-2221-E-216-008 is greatly acknowledged.

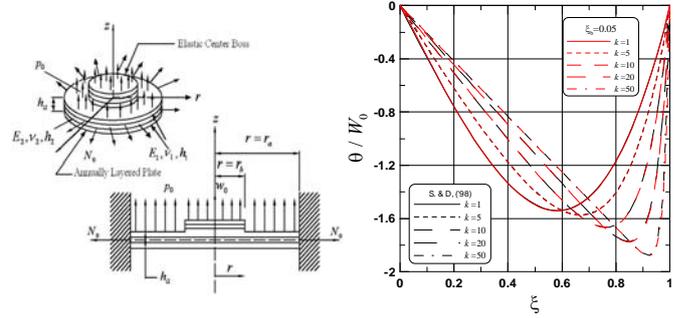


Fig. 1 Prob. Schematics Fig. 2 Slope for a NUB Plate

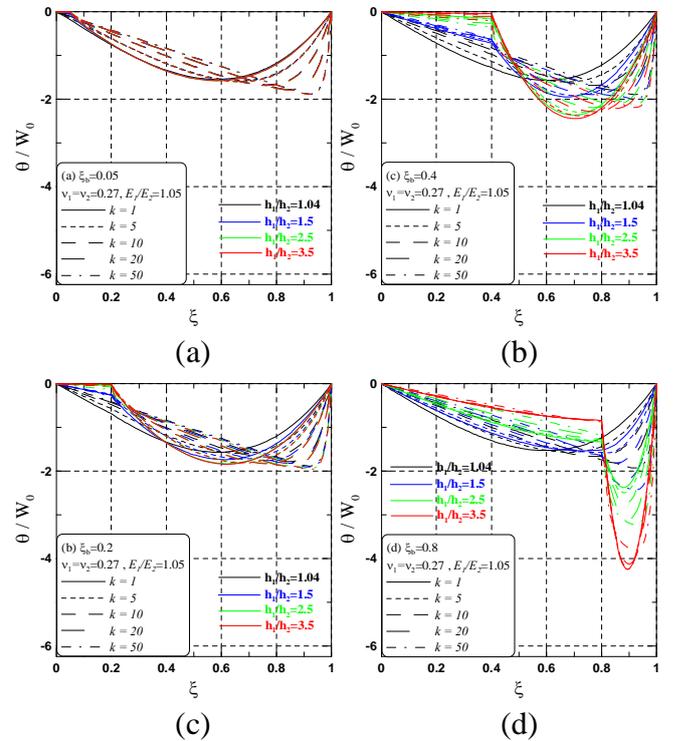


Fig. 3 Slopes for NSL Plates & Various Boss Sizes.

#### VI. References

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