VARIATIONAL BOUNDS BASED ON EXTENDED H-S PRINCIPLES

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Introduction

Classical Hashin-Shtrikman bounds (abbreviated as H-S principles) are based on assumption that all phases behave purely elastic (Hashin & Shtrikman 1962). The principles have been extended in (Prochazka & Sejnoha 2004) introducing eigenparameters, which can stand for plastic strain or relaxation stresses. In classical approach of Hashin and Shtrikman the energies of entire structure were compared with local energies (on the micro scale level), and so was the idea of the procedure in-volving eigenparameters. A simple example will be studied and bounds on material properties will be provided. Fundaments of bounds based on extended H-S principles were derived in (Toman, 2007). As the extended variational bounds can be derived in similar way as was the approach proposed by Hashin and Shtrikman, no computational procedure is needed for the estimates.

Brief overview of Extended H-S principles

Extended H-S principles are generalization of classical H-S principles, involving eigenparameters in the formulation.

Let the area $\Omega \in R_3$ with the boundary Γ is given, describing a shape of composite body. Furthermore, let us assume that displacements $\bar{u}_i$, $i = 1,2,3$ are prescribed in the whole boundary, i.e $\Gamma \equiv \Gamma_u$. The procedure is divided into two steps. In the first step the quantities $\boldsymbol{u}^o$, $\boldsymbol{p}^o$, $\boldsymbol{\varepsilon}^o$ and $\boldsymbol{\sigma}^o$ for a homogenous and isotropic body are to be determined. The static equations apply inside the body Ω and Hook's law with the stiffness matrix $\boldsymbol{L}^o$ can be applied as well:

$$\sigma_{ij}^o = L_{ijkl}^o \varepsilon_{kl}^o \text{ in } \Omega , \quad u_i^o = \bar{u}_i \text{ on } \Gamma \qquad (1)$$

The quantities for this state are denoted by upper index *o*. In the sense of standard procedure we can obtain displacements $\boldsymbol{u}^o$ and surface forces $\boldsymbol{p}^o$. Depending on $\boldsymbol{u}^o$, the deformation tensor $\boldsymbol{\varepsilon}^o$ or the stress distribution $\boldsymbol{\sigma}^o$ can be calculated. These quantities are considered to be known in the further text.

In the second step the formulation leading to a solution of the quantities $\boldsymbol{u}$, $\boldsymbol{p}$, $\boldsymbol{\varepsilon}$ and $\boldsymbol{\sigma}$ for a general inhomogeneous and anisotropic body follows. The geometrically identical body with the same boundary displacements $\bar{\boldsymbol{u}}$ from the first step is taken into account. Then, the real displacements $\boldsymbol{u}$, deformations $\boldsymbol{\varepsilon}$ and stresses $\boldsymbol{\sigma}$ are unknown and Hook's law can be applied for them, involving eigenstresses $\boldsymbol{\lambda}$, or eigenstrains $\boldsymbol{\mu}$:

$$\sigma_{ij} = L_{ijkl} \varepsilon_{kl} + \lambda_{ij} = L_{ijkl} (\varepsilon_{kl} \quad \mu_{ij}) \text{ in } \Omega ,$$
$$u_i = \bar{u}_i \text{ on } \Gamma \qquad (2)$$

Furthermore let us introduce primed system as:

$$u' = u_i - u_i^o , \varepsilon_{ij}' = \varepsilon_{ij} - \varepsilon_{ij}^o , \sigma_{ij}' = \sigma_{ij} - \sigma_{ij}^o \text{ in } \Omega ,$$
$$u_i' = 0 \text{ on } \Gamma \qquad (3)$$

Defining the polarization tensor $\boldsymbol{\tau}$ as:

$$\tau_{ij} - [L_{ijkl}] \varepsilon_{kl} - \lambda_{ij} = 0, \quad [\boldsymbol{L}] = \boldsymbol{L} - \boldsymbol{L}^o \text{ in } \Omega$$
$$\text{or} \qquad (4)$$
$$C_{ijkl} (\tau_{kl} - \lambda_{kl}) - \varepsilon_{ij} = 0, [L_{ijrs}] C_{rskl} = I_{ijkl}$$

where $\boldsymbol{I}$ is the unit tensor of the fourth order. Note that in comparison to the classical H-S principles the polarization tensor is extended by eigenparameters, either by eigenstress or by eigenstrain. This is obvious advantage of this approach. The density of deformation energy of the extended primary H-S principle is described with the following formula:

$$U(\boldsymbol{\tau}, \boldsymbol{\varepsilon}') = U^o \quad \frac{1}{2} \int_\Omega \{ C_{ijkl} (\tau_{ij} - \lambda_{ij})(\tau_{kl} - \lambda_{kl})$$
$$2 \tau_{ij} \varepsilon_{ij}^o - \varepsilon_{ij}' \tau_{ij} - M_{ijkl} \lambda_{ij} \lambda_{kl} \} \mathrm{d}\boldsymbol{x} \qquad (5)$$
$$U^o = \frac{1}{2} \int_\Omega \sigma_{ij}^o \varepsilon_{ij}^o \mathrm{d}\Omega = \frac{1}{2} \int_\Omega L_{ijkl}^o \varepsilon_{ij}^o \varepsilon_{kl}^o \mathrm{d}\Omega .$$

It appears that similarly to H-S principles the functional (5) reaches its absolute maximum $(\delta^2 U \le 0)$ if $[\boldsymbol{L}]$ is positively definite, and it reaches its absolute minimum $(\delta^2 U \ge 0)$ if $\boldsymbol{C} + \boldsymbol{M}^o$ is negatively semidefinite. The proof for the conditions of maximum and minimum can be found in (Toman, 2007). Note here that in the original paper by Hashin and Shtrikman the condition of minimum is not quite correct, as they suppose that only $\boldsymbol{C}$ has to be negatively semidefinite and this is not sufficient condition.

Variational bounds

Here we partly use results of (Toman, 2007). Let us come out from the following functional for the Mises hypothesis:

$$\inf_{u\in U} \int_{\Omega} \frac{1}{2} K_0 (\mathrm{div}\boldsymbol{u})^2 + \Phi(\varepsilon^d(\boldsymbol{u}))\ \mathrm{d}x \quad , \qquad (6)$$

where

$$\Phi(\varepsilon^d) = G_0 e^d_{ij} e^d_{ij}, \text{ if } \quad e^d_{ij} e^d_{ij} \le \frac{k^2}{2(G_0)^2}$$

$$\Phi(\varepsilon^d) = G_0 e^d_{ij} e^d_{ij} + k(\sqrt{(2e^d_{ij} e^d_{ij})} - \frac{k}{2G}) \text{ otherwise}$$

The estimates for the bounds on the shear modules are recorded as:

$$G_1^* \le G^* \le G_2^*,$$

$$G_1^* = G_1 + \frac{1}{2}\frac{B_1}{1+\beta_1 B_1} - B^p, \quad G_2^* = G_n + \frac{1}{2}\frac{B_n}{1+\beta_n B_n} - B^p,$$

$$B^p = v_p m \frac{G_p}{2}, \quad \beta_1 = \frac{3(K_1 + 2G_1)}{5G_1(3K_1 + 4G_1)},$$

$$\beta_n = -\frac{3(K_n + 2G_n)}{5G_n(3K_n + 4G_n)}, \qquad B_1 = \sum_{r=2}^{n} \frac{v_r}{\frac{1}{2(G_r - G_1)} - \beta_1},$$

$$B_n = \sum_{r=1}^{n-1} \frac{v_r}{\frac{1}{2(G_r - G_n)} - \beta_n}$$

and m is the rate of plasticity, v_r are volume fractions of constituents, K, G are the bulk and shear modules.

Applications to the 2-Phase Material

Assuming ideally elastic-plastic behavior by Mises and considering 2-phase composite, the bounds estimates provide:

$$G_1^* = G_1 + \frac{1}{2}\frac{B_1}{1+\beta_1 B_1} - m\frac{G_1}{2} v_1,$$

$$G_2^* = G_2 + \frac{1}{2}\frac{B_2}{1+\beta_2 B_2} - m\frac{G_1}{2} v_1,$$

$$\beta_1 = \frac{3(K_1 + 2G_1)}{5G_1(3K_1 + 4G_1)}, \quad \beta_2 = \frac{3(K_2 + 2G_2)}{5G_2(3K_2 + 4G_2)},$$

$$B_1 = \frac{v_2}{\frac{1}{2(G_2 - G_1)} - \beta_1}, \quad B_2 = \frac{v_1}{\frac{1}{2(G_1 - G_2)} - \beta_2}$$

The material constants are in GPa: $K_1 = 395$, $K_2 = 959$, $G_1 = 23.7$, $G_2 = 647.8$. The pictures Fig. 1 and Fig. 2 show the function of bounds on the rate of plasticity under various volume fractions.

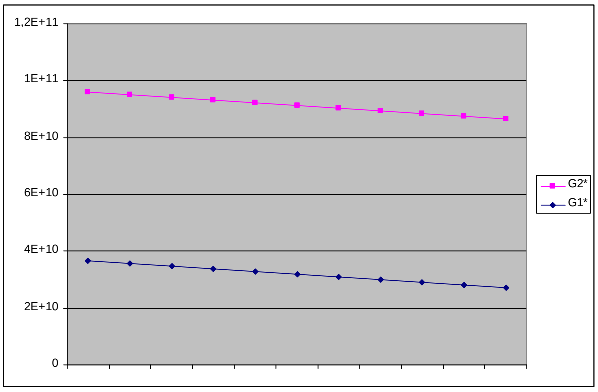

Fig. 1 Function of bounds G_1^*, G_2^* on m for

$v_1 = 0.8, \quad v_2 = 0.2$

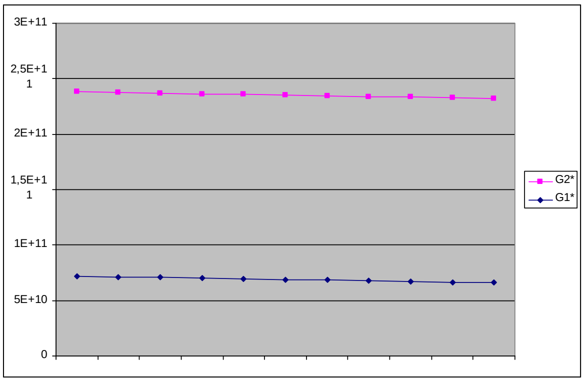

Fig. 2 Functions of bounds G_1^*, G_2^* on m for

$v_1 = 0.5, \quad v_2 = 0.5$

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References

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