EIGENFREQUENCIES OF 2D THICK-WALLED COMPOSITE ARCHES

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Introduction

In this work laminated arches are solved under condition that isotropic layers are considered. Mathematical apparatus is briefly discussed and formulas needed for programming on computers are derived. The focus here is concentrated on a problem of impact load, which can be simulated by developing time coordinate into the Fourier series. The solution is formulated for simply supported or clamped layered arch. A semi-analytic solution is used for simply supported arch, and the conditions for clamped edges are taken from a well known Lechnitski's book on Theory of plates, [1]. The boundary conditions are identified by selection of sine or cosine series applied to different directions of displacements. Two-dimensional example is solved.

Behavior in one lamina

This section is focused on one lamina from laminate cylindrical segment (arch). We start with introducing polar coordinates $0r\theta$, which are derived from Cartesian coordinates $0xy$. Components of the displacement vector are $u = u_r, v = u_\theta$, where u_r is the displacement in the radial direction and u_θ is the displacement in the circumferential (hoop) direction, and the plain strain state is considered. The cylindrically isotropic structure is described by Hooke's law in each lamina:

$$\boldsymbol{\sigma} = \boldsymbol{L}\boldsymbol{\varepsilon}, \text{ or in vector notation}$$

$$\sigma_i = L_{ij}\varepsilon_j, i, j = r, \theta \text{ and } \sigma_{r\theta} = G_{r\theta}\varepsilon_{r\theta} \quad (1)$$

where the stress tensor $\boldsymbol{\sigma}$ is related with the strain tensor $\boldsymbol{\varepsilon}$ through the material stiffness matrix, and $L_{ij} = L_{ji}, G_{r\theta}$ are stiffness coefficients, $L_{rr} = L_{\theta\theta}$, $L_{rr} - L_{r\theta} = 2G_{r\theta}$. From these assumptions it follows that only two material coefficients describe each lamina. In the above description the number of layer is dropped out.

The kinematical equations are written in polar coordinates as:

$$\varepsilon_r = \frac{\partial u}{\partial r}, \quad \varepsilon_\theta = \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{u}{r}, \quad \varepsilon_{r\theta} = \frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r} \quad (2)$$

The Hooke law can now be written as:

$$\begin{aligned}
\sigma_r &= L_{rr}\varepsilon_r + L_{r\theta}\varepsilon_\theta = L_{rr}\frac{\partial u}{\partial r} + L_{r\theta}\left(\frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{u}{r}\right)\\
\sigma_\theta &= L_{r\theta}\varepsilon_r + L_{\theta\theta}\varepsilon_\theta = L_{r\theta}\frac{\partial u}{\partial r} + L_{\theta\theta}\left(\frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{u}{r}\right) \quad (3)\\
\sigma_{r\theta} &= G_{r\theta}\varepsilon_{r\theta} = G_{r\theta}\left(\frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r}\right)
\end{aligned}$$

Two equations of equilibrium provide relations among three components of stress tensor:

$$\begin{aligned}
&\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\left(\frac{\partial \sigma_{r\theta}}{\partial \theta} + \sigma_r - \sigma_\theta\right) - \rho\frac{\partial^2 u_r}{\partial t^2} = 0,\\
&\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r}\left(\frac{\partial \sigma_\theta}{\partial \theta} + 2\sigma_{r\theta}\right) - \rho\frac{\partial^2 u_\theta}{\partial t^2} = 0
\end{aligned} \quad (4)$$

where ρ is the mass density of the lamina under account, t is the time.

We solve now the declared problem on one layer (lamina) in terms of the semi-analytical method. For this we introduce sine and cosine series and a coefficient α, which describes the position of a concrete term in the series. Expressing α_ℓ as $\alpha_\ell = m.\ell$, where positive integer ℓ is the number in the term in the Fourier series and $m = \dfrac{\pi}{\beta}$.

Expanding two components of displacements into sine and cosine series and denoting the ℓ-th term depending only on r by U_r^ℓ and U_θ^ℓ yields

$$\begin{aligned}
u_r^\ell(r,\theta,t) &= U_r^\ell(r)\cos\alpha_\ell\theta\exp(\mathrm{i}\omega t),\\
u_\theta^\ell(r,\theta,t) &= U_\theta^\ell(r)\sin\alpha_\ell\theta\exp(\mathrm{i}\omega t)
\end{aligned} \quad (5)$$

where $U_r^\ell \equiv U^\ell$ and $U_\theta^\ell \equiv V^\ell$ are unknown functions of r which need to be determined from the equations of equilibrium. In particular, substituting (1) into the stress-strain relations gives stresses which, when substituted into the equations of equilibrium in cylindrical coordinates, provide the following equations for evaluation of U^ℓ and V^ℓ.

Components of strains, (2), and stresses, (3), are then expressed using (5). Substituting the result into the first static equation (4) one gets:

$$U'' + \frac{U'}{r} + A_1\frac{U}{r^2} + A_2\frac{V'}{r} + A_3\frac{V}{r^2} = 0 \quad (6)$$

where

$$A_1 = \frac{(L_{rr} + \alpha^2 G_{r\theta})}{L_{rr}} + (\bar{r}\omega)^2, \; A_2 = \frac{\alpha(L_{r\theta} + G_{r\theta})}{L_{rr}},$$

$$A_3 = \frac{\alpha(L_{rr} + G_{r\theta})}{L_{rr}}$$

The second equation of equilibrium is the following:

$$V'' + \frac{V'}{r} + B_1 \frac{V}{r^2} + B_2 \frac{U'}{r} + B_3 \frac{U}{r^2} = 0 \qquad (7)$$

where:

$$B_1 = \frac{(G_{r\theta} + \alpha^2 L_{rr})}{G_{r\theta}} + (\bar{r}\omega)^2, \; B_2 = \frac{\alpha(L_{r\theta} + G_{r\theta})}{G_{r\theta}},$$

$$B_3 = \frac{\alpha(L_{rr} + G_{r\theta})}{G_{r\theta}}$$

and $\bar{r}$ is the average of the radii in the trial lamina.

The solution of simultaneous equations (6) and (7) can be done using the substitution: $r = \exp(\lambda)$. Then the above differential equations are solved in standard way to get roots λ of characteristic equation of the fourth order to the fundamental solutions: $r^{\lambda_1}, r^{\lambda_2}, r^{\lambda_3}, r^{\lambda_4}$. The characteristic equation provides roots:

$$\lambda^4 + (A_1 + B_1 - A_2 B_2)\lambda^2 + A_1 B_1 - A_3 B_3 = 0, \; \lambda^4 - b\lambda^2 + c = 0 \qquad (8)$$

where $\lambda_{1,2,3,4} = \sqrt{\frac{b \quad \sqrt{b^2 - 4c}}{2}} = \quad (1 \quad \alpha)$.

The fundamental solution for unknown amplitudes of displacements U and V follows as:

$$U^\ell = \{(A_2\lambda_1 + A_3)A^\ell r^{\lambda_1} + (A_2\lambda_1 - A_3)B^\ell r^{-\lambda_1} +$$
$$+ (A_2\lambda_2 + A_3)C^\ell r^{\lambda_2} + (A_2\lambda_2 - A_3)D^\ell r^{-\lambda_2}\}\cos\alpha_\ell\theta$$
$$V^\ell = \{(B_2\lambda_1 + B_3)A^\ell r^{\lambda_1} + (B_2\lambda_1 - B_3)B^\ell r^{-\lambda_1} +$$
$$+ (B_2\lambda_2 + B_3)C^\ell r^{\lambda_2} + (B_2\lambda_2 - B_3)D^\ell r^{-\lambda_2}\}\sin\alpha_\ell\theta \qquad (9)$$

The last step in the computation in one lamina is expressing the strains and stresses. It is enough to use (9) and substitute them in the expression (2) to obtain strains and to (3) to get stresses.

Entire structure

Introducing $r = a_k$ and $r = b_k$ to be the radii of boundaries of the lamina k, yields the interfacial and boundary conditions ($P_a^\ell = 2\pi\sigma_r^\ell a_k$, $P_b^\ell = 2\pi\sigma_r^\ell b_k$, $S_a^\ell = 2\pi\sigma_{r\theta}^\ell a_k$ $S_a^\ell = 2\pi\sigma_{r\theta}^\ell b_k$):

for

$r = a_k = b_{k-1}$:

$P_a^\ell(k) + P_b^\ell(k-1) = P_k^\ell$, $S_a^\ell(k) + S_b^\ell(k-1) = S_k^\ell$,

$U_a^\ell(k) = U_b^\ell(k-1)$, $V_a^\ell(k) = V_b^\ell(k-1)$

On the boundaries, for $r = a$ and $r = b$ the external load has to be developed in the Fourier series and the coefficients in it must be in a compliance with the adjacent layers. The resulting linear algebraic system of equations can be stored as:

$$\boldsymbol{P} = \boldsymbol{K}(\omega)\boldsymbol{U} \qquad (10)$$

where the left hand side is the vector describing the interfacial radial and shear tractions, $\boldsymbol{K}$ is the stiffness and mass matrix of the structure involving the eigenfrequency ω, $\boldsymbol{U}$ means the vector of radial and hoop displacements on the interfaces.

Examples and conclusions

In order to verify the accuracy of the suggested approach attained on an arch with the following dimensions and material properties are: the length l = 1 m, the external radius r = 2 m, the thickness = 5 mm, β = 0.5 rad, E = 200 GPa, ρ = 7833 kg/m^3, ν = 0.29. Always one layer is different form the others, defined above. The difference consists in elastic modulus E = 20 GPa. According to the position of this particular layer the eigenfrequencies are depicted in Fig. 1. From this picture it is seen that the largest eigenfrequencies are attained if the particular layer is positioned in lower laminas, as the numbering is from lower to upper layer. From couple of examples this result can be generalized.

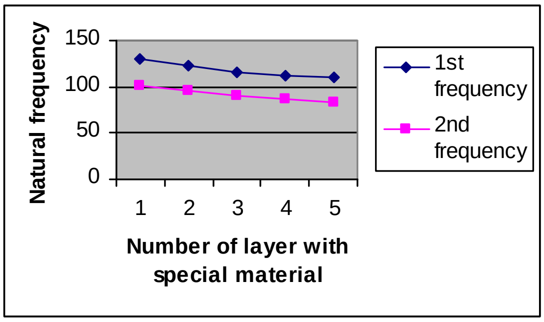

Fig. 1 The first and the second eigenfrequencies

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References

[1] Lechnitskii, S.G. Anisotropic plates. Gordon and Briearch science publishers, New York, London, Paris, 1968