Multiple Solutions on a Ball for a Generalized Lane–Emden Equation

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Abstract

I examine the nonlinear ordinary differential equation corresponding to radial solutions of the Dirichlet problem

$$\nabla \cdot (|x|^{\alpha} |\nabla u|^{p-2} \nabla u) + |x|^{\beta} |u|^{q-1} u = 0 \qquad (1)$$

$$u|_{\partial B^n} = 0$$

for $q > 1, 1 < p \leq 2, \alpha, \beta \geq 0$, where $B^n = \{x \in R^n : |x| \leq 1\}$. I establish the existence and non-uniqueness of infinitely many radially symmetric solutions, controlled by a Sobolev critical exponent, to the generalized form of the Lane-Emden equation (GLE). I prove the non-existence of solutions for $q+1 \geq p^*$, $p^* = \frac{p(n+\beta)}{n+\alpha-p}$. I also prove that the class of all nontrivial solutions are bounded below for $p < q+1 < p^*$, $1 < p \leq 2, q > 1, \alpha, \beta \geq 0$ by a constant dependent on p, n, α, β.

1 Introduction

Equation (1) serves as a weighted generalization of the Lane-Emden equation $\Delta u + u^q = 0$, which has attracted great interest in the literature and has undergone extensive research due to its frequent appearance in mathematics and astrophysics. The Lane-Emden Dirichlet problem on a ball of radius $r > 0$, $u|_{\partial B^n} = 0$ for $u > 0$, has been studied extensively by Serrin. He proved that only for $q < \frac{n+2}{n-2}$ there exist radial positive solutions to the Lane-Emden equation and the nonexistence of such solutions for $q \geq \frac{n+2}{n-2}$, where $\frac{n+2}{n-2}$ is known as the Sobolev critical exponent. In 1973, Henon proposed a general form of the Lane-Emden equation, involving the weight function $|x|^l$, to model rotating stellar systems, referred to as the Emden-Fowler equation,

$$\Delta u + |x|^l u^q = 0 \qquad (2)$$

$$u > 0 \text{ in } \Omega \text{ and } u = 0 \text{ on } \partial\Omega.$$

where Ω is a bounded smooth domain in R^n. It was shown that the Sobolev critical exponent in this case is changed to $q = \frac{n+2+2l}{n-2}$ and classical radial solutions on a ball exist for $q < \frac{n+2+2l}{n-2}$. Several other generalizations of the Lane-Emden equation followed to cover more applications arising in geometry, physics, astrophysics and Engineering, [1], [2], [3]. In particular, (1) appears as an application in fluid mechanics as a physical phenomena related to equilibrium of anisotropic media which possible are perfect insulators. Let τ be the sheer stress and $\nabla_p u (= |\nabla u|^{p-2} \nabla u, p > 1)$ is the velocity gradient, then these quantities obey a relation of the form $\tau = a(x)\nabla_p u(x)$. The resulting equations of motion then involve the nonlinear, inhomogeneous expression $\nabla \cdot (a\nabla_p u)$, which reduces to $a \cdot (\nabla_p u)$ for a being a constant, [4].

2 Preliminaries

I examine bounded solutions, in particular, let $u(0) = \sigma, |\sigma| < \infty, u_r(0) = 0, u(1) = 0$, $u_r(1) = -\delta, -\delta \in (-\infty, \infty)$. Letting $E(r) = \frac{p-1}{p} r^{\alpha} |u_r|^p + \frac{1}{q+1} r^{\beta} |u|^{q+1}$ represent the energy function of solutions, then $E_1(r,\sigma) = r^{-\beta} E(r,\sigma)$ is a monotone decreasing function for a fixed value of σ, such that,

Lemma 2.1. *$\frac{d}{dr} E_1(r,\sigma) \leq 0$ for $\alpha, \beta \geq 0, 1 < p \leq 2, q > 1, u(0) = |\sigma| < \infty, u_r(0) = 0$, $r \geq 0$.*

Lemma 2.2. *Let $u(r)$ satisfy (1), $u(0) = \sigma$, $u_r(0) = 0, \alpha, \beta \geq 0, 1 < p \leq 2$, $p^* = \frac{p(n+\beta)}{n+\alpha-p}$, then for $r \in [0,1]$ and $\lambda = \frac{1}{q+1} - \frac{1}{p^*}$,*

$$\frac{1}{n+\beta} r^n E(r,\sigma) = \lambda \int_0^r s^{n+\beta-1} |u|^{q+1} ds - \frac{1}{p^*} r^{n+\alpha-1} |u_r|^{p-2} u_r u.$$

3 Non-existence result

Theorem 3.1. *For $\alpha, \beta \geq 0, 1 < p \leq 2, q > 1, u(0) = |\sigma| < \infty, u_r(0) = 0$, $q+1 \geq p^*, p^* = \frac{p(n+\beta)}{\alpha+n-p}$, no radial solutions exist for the generalized Lane-Emden equation (1).*

I also establish the nonexistence of more general solutions for $q+1 \geq p^*$ by applying Pohozaev's identity to the GLE equation (1).

Corollary 3.1. *For $\alpha, \beta \geq 0, 1 < p \leq 2, q > 1, q+1 \geq p^*$, $p^* = \frac{p(n+\beta)}{\alpha+n-p}$, there exist no solutions to the generalized Lane-Emden equation over any domain which is smooth and star-shaped with respect to the origin.*

4 Existence and nonuniqueness result.

Proving the existence of infinitely many radially symmetric solutions to the boundary value problem

$$(p-1)r^{\alpha} |u_r|^{p-2} u_{rr} + (\alpha+n-1)r^{\alpha-1} |u_r|^{p-2} u_r + |r|^{\beta} |u|^{q-1} u = 0 \quad (3)$$

$$u(1) = u_r(0) = 0 \quad (4)$$

is done using the shooting argument which relates (4.1), (4.2) to the initial value problem (4.1) with initial conditions (4.3),

$$u(0) = \sigma, u_r(0) = 0 \quad (5)$$

The value of σ is chosen in a manner that the solution to (4.1), (4.3) also satisfies $u(1) = 0$. This solution exists for $1 < p \leq 2$, $\alpha, \beta \geq 0$, $p < q+1 < p^*$ and is unique for $r \in [0,1]$, as will be shown next. Using the shooting argument I will be able to show that there exists a sequence $\{\sigma_n\}$, $n = 1, 2, 3, ...$ of values of σ, each of which gives rise to a corresponding solution of (4.1), (4.2). Thus the σ_n parametrize an infinite sequence of solutions, $\{u_n(r)\}$, $n =$ 1,2,3, ... to the boundary value problem (4.1), (4.2), with $\sigma_n = u_n(0) \to \infty$ as $n \to \infty$. I start by proving the existence of a unique radial solution to the corresponding initial value problem (4.1), (4.3). This is done by defining an integral operator and proving the existence of a unique fixed point of this operator. Next, I prove that radial bounded solutions approach the origin with zero slope for $\beta - \alpha + 1 > 0$, which in turn implies that $\theta = \frac{p+\beta-\alpha}{q+1-p}$ is positive for $1 < p \leq 2$, $q > 1$, $\alpha, \beta \geq 0$.

Theorem 4.1. *For $\beta - \alpha + 1 > 0, 1 < p \leq 2, q > 1$, radial solutions of (1.1) satisfying the boundary condition in (1.2) approach the origin with zero slope such that $u_r(0) = 0$.*

Corollary 4.1. *Let $u(r)$ satisfy (1.1), (4.2), (4.3) and $\alpha, \beta \geq 0, 1 < p \leq 2$,*
$q+1 \neq p, \theta = \frac{p+\beta-\alpha}{q+1-p} < 0$, then

$$E(r,\sigma) = \frac{p-1}{p} r^{\alpha} |u_r|^p + \frac{1}{q+1} r^{\beta} |u|^{q+1} \to \infty \text{ as } |\sigma| \to \infty$$

uniformly for $r \in [0,1]$.

Theorem 4.2. $\Theta(1,\sigma) \to \infty$ *as* $\sigma \to \infty$

Theorem 4.3. *Let $\alpha, \beta \geq 0, 1 < p \leq 2, p < q+1 < p^* = \frac{p(n+\beta)}{\alpha+n-p}$. Then (1) has infinitely many radially symmetric solutions lying in*
$W^{1,p}(B^n, |x|^{\alpha}) \cap L^{q+1}(B^n, |x|^{\beta})$.

5 Bounded solutions

Theorem 5.1. *General solutions for the generalized Lane-Emden equation (1) are bounded below by a constant $C = C(p,n,\alpha,\beta)$ for $\alpha, \beta \geq 0, 1 < p \leq 2$, $p < q+1 < p^*$.*

I also prove in a similar manner the same property for radial solutions of (1) for $\alpha, \beta \geq 0, 1 < p \leq 2, p < q+1 < p^*$.

Corollary 5.1. *Radial solutions for the generalized Lane-Emden equation (1) are bounded below for $\alpha, \beta \geq 0, 1 < p \leq 2, p < q+1 < p^*$ by a constant $C(n,p,\alpha,\beta)$.*

References

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