Piecewise Lognormal Approximation to Lognormal Sum of Leakage Current in Nano-scale CMOS

FANG WANG
Faculty of Applied Science, Memorial University of Newfoundland, NL, Canada A1B 3X5

I. Introduction

In the nano-scale CMOS circuit, the continued scaling of transistors and shrinking threshold voltages leads to drastically increase of leakage power and therefore leakage current has emerged as a stringent constraint in modern circuit design. Actual measurements showed that leakage of a single gate, mainly consisting of sub-threshold leakage and gate tunneling leakage, can be well modeled as a lognormal distribution [1]; as a result the full-chip leakage current can be computed by summing up leakage current of each gate, that is, a sum of lognormal random variables, see e.g. [2] and the references therein. Since the characteristic function of a lognormal distribution has no closed form, lognormal sum distribution has to be numerically calculated. However, due to the extremely slow decreasing tail of lognormal distribution, numerical computing of a lognormal sum distribution is of significant challenges and therefore lognormal sum distributions have received considerable attention during the past five decades, see e.g. [3], [4] and the references therein.

On lognormal probability paper, the cumulative distribution function (CDF) plotting of a lognormal (LN) distribution is a straight line; however, numerical examples all showed that the CDF of a LN sum is a concave curve on lognormal probability paper, which means LN approximation can only capture part of the LN sum. For this reason, many researchers resort to non-lognormal approximations, see e.g. [3]. These non-lognormal approximations can match the CDF of a lognormal sum with very high accuracy, e.g. 10^{-6}, but their computations can be so complicated and sensitive that they are not easy to use in practice. Since computing the exact distribution of a lognormal sum is very challenging, in practice lognormal is usually used to approximate lognormal sum distributions for simplicity. In this work, we first compare three widely applied lognormal approximations and then propose a piecewise lognormal approximation. Numerical examples are provided to illustrate the effectiveness of the proposed scheme.

II. Lognormal Approximations

A random variable X is said to have the lognormal distribution with parameters μ, σ, denoted by $LN(\mu, \sigma)$ if its probability density function (PDF) is given by

$$f(x; \mu, \sigma) = \begin{cases} \frac{1}{\sqrt{2\pi}\sigma x} \exp\left(-\frac{(\log(x)-\mu)^2}{2\sigma^2}\right), & x > 0 \\ 0, & x \leq 0, \end{cases} \tag{1}$$

If $X \sim LN(\mu, \sigma)$, then $10 \log_{10} X \sim \mathbf{N}(\mu, \sigma^2)$ with unit decibels (dB), where μ and σ are called the *scale* and *spread* parameters, respectively.

If X has the lognormal distribution $LN(\mu, \sigma)$, its CDF is $\Phi(\frac{\log(x)-\mu}{\sigma})$ where Φ denotes the CDF of the standard normal distribution. Direct calculation gives its k-th order moment

$$\mathbf{E}[X^k] = exp(k\mu + k^2\sigma^2/2). \tag{2}$$

Let $X_1, X_2, \cdots, X_n$ be n random variables having lognormal distributions and let $S_n = X_1 + X_2 + \cdots + X_n$. The central problem of LN approximation of the lognormal sum S_n is to find a random variable Y with the distribution $LN(\mu_y, \sigma_y)$ to approximate S_n, i.e. to find appropriate parameters σ_y and μ_y so that Y captures the desired part of the distribution of S_n. Next, we present three most commonly applied LN approximations.

Wilkinson's LN approximation is the earliest and most popular one. Let u_1 and u_2 be the first and second moment of S_n. Wilkinson's approximation [4] computes μ_y, σ_y by matching the first and second order moments, that is, assume $exp(\mu_y + \sigma_y^2/2) = u_1,\ exp(2\mu_y + 2\sigma_y^2) = u_2$, yielding

$$\mu_y = 2\log(u_1) - \log(u_2)/2,$$
$$\sigma_y = \log(u_2) - 2\log(u_1)$$

Schwartz-Yeh's approximation (SY) [6] is much more involved, which first derives the first two moments of the sum of two lognormal random variables, and then a recursive procedure is applied to extend the approach to a larger number of summands. For the details of the SY approximation, we refer the reader to [6].

Filho, Cardieri and Yacoub (FCY) [5] proposed an approximation by matching the the first and second moments of $1/S_n$ and $1/Y$, that is,

$$E[1/Y] = E[1/S_n],\ E[1/Y^2] = E[1/S_n^2] \tag{3}$$

which combined with (2) yields

$$\mu_y = \log(E[1/S_n^2])/2 - 2\log(E[1/S_n]), \tag{4}$$
$$\sigma_y = \sqrt{\log(E[1/S_n^2]) - 2\log(E[1/S_n])} \tag{5}$$

By numerical examples, we find that Wilkinson's approximation captures the right tails of the LN sum distributions, SY approximation captures the main body but not the tails; FCY approximation captures the left tails of the LN sum very closely.

III. Piecewise Lognormal Approximation

Let $F(x;\mu_1,\sigma_1)$ and $F(x;\mu_2,\sigma_2)$ be two lognormal distributions with parameters (μ_1,σ_1) and (μ_2,σ_2), respectively. Then, direct computation yields that

$$x=\exp\left(\frac{\mu_1\sigma_2-\mu_2\sigma_1}{\sigma_2-\sigma_1}\right) \tag{6}$$

is the unique solution of

$$F(x;\mu_1,\sigma_1)=F(x;\mu_2,\sigma_2). \tag{7}$$

Let $S_n=X_1+X_2+\cdots+X_n$ be a lognormal sum distribution and let $LN(\mu_w,\sigma_w)$, $LN(\mu_{sy},\sigma_{sy})$ and $LN(\mu_{fcy},\sigma_{fcy})$ denote, respectively, its Wilkinson's, SY, and FCY approximations. Their CDFs meet at

$$x_1^*=exp\left(\frac{\mu_{fcy}\sigma_{sy}-\mu_{sy}\sigma_{fcy}}{\sigma_{sy}-\sigma_{fcy}}\right),$$
$$x_2^*=exp\left(\frac{\mu_{w}\sigma_{sy}-\mu_{sy}\sigma_{w}}{\sigma_{sy}-\sigma_{w}}\right),$$
$$x_3^*=exp\left(\frac{\mu_{fcy}\sigma_{w}-\mu_{w}\sigma_{fcy}}{\sigma_{w}-\sigma_{fcy}}\right).$$

Numerical examples show that the FCY, SY and Wilkinson's approximations can closely capture, respectively, the left tail, main body and right tail of a lognormal sum distribution. So, a better solution is to approximate the lognormal sum distribution piecewisely, that is, approximate the lognormal sum distribution by FCY approximation in $(0,x_1^*)$, by SY approximation in $[x_1^*,x_2^*)$ and by the Wilkinson's approximation in $[x_2^*,\infty)$. The CDF of the three-piecewise lognormal approximation can be given by

$$F_{pw3}(x)=\begin{cases} F(x;\mu_{fcy},\sigma_{fcy}), \text{ if } x\le x_1^* \\ F(x;\mu_{sy},\sigma_{sy}), \text{ if } x_1^*<x\le x_2^* \\ F(x;\mu_{w},\sigma_{w}), \text{ if } x>x_2^* \end{cases} \tag{8}$$

Figure 1 plots the CDF of $LN(0,6)+LN(0,12)$, its LN approximations and piecewise lognormal approximation, showing the piecewise lognormal approximation is much closer to the real lognormal sum than the lognormal approximations. For a more general example, let us consider the sum of 8 independent LN distributions $LN(0,i), i=5,6,\cdots,12(dB)$. Figure 2 plots the empirical CDF of the lognormal sum distribution and its piecewise LN approximation, showing the piecewise LN approximation works very well.

IV. Conclusion

In this work, we proposed a piecewise lognormal approximations. Numerical examples showed the proposed piecewise lognormal approximation has much better performance than the lognormal approximations. As future work, we will apply the proposed lognormal sum approximation to low power digital circuit design.

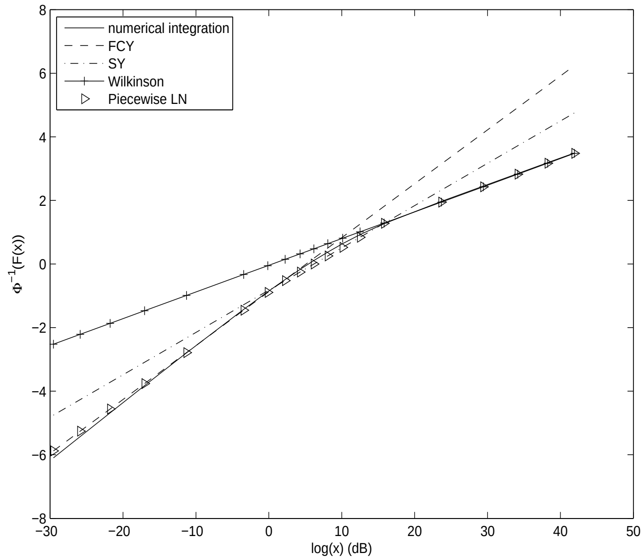

Fig. 1. Results of LN Approximation to $LN(0,6)+LN(0,12)$

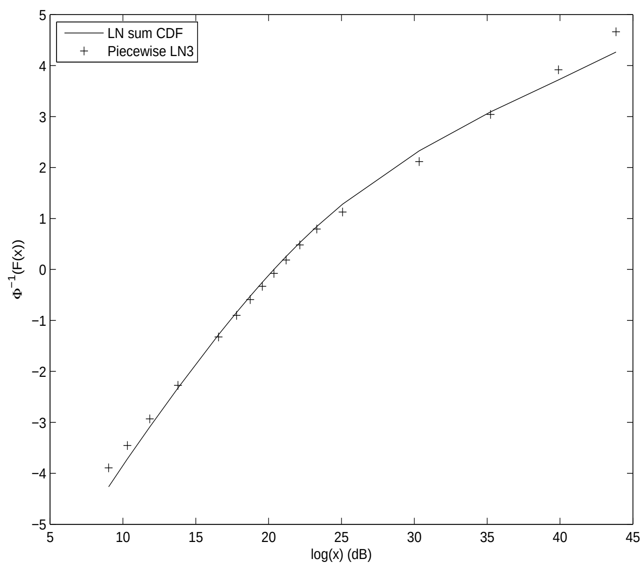

Fig. 2. Piecewise LN Approximations to $LN(0,5)+LN(0,6)+\cdots+LN(0,12)$

References

[1] M. Anis and M. H. Aburahma, "Leakage current variability in nanometer technologies," *Proc. of IEEE SoC for real time applications*, pp. 60- 63, 2005.

[2] C. D'Agostino, P. Flatresse, E. Beigne and M. Belleville, "Statistical leakage modeling in CMOS logic gates considering process variations," *Proc. of IEEE ICICDT 2008*, pp. 301-304, 2008.

[3] Z. Liu, J. Almhana, and R. McGorman. Approximating Lognormal Sum Distributions with Power Lognormal Distributions, IEEE Trans on VT, vol. 57, No. 4, pp. 2611-2617, 2008.

[4] L. F. Fenton, "The sum of log-normal probability distributions in scatter transmission systems," *IRE Trans. Commun.*, Vol. 8, pp. 57-67, 1960.

[5] J.C.S. Santos Filho, P. Cardieri and M.D. Yacoub, "Simple accurate lognormal approximation to lognormal sums," *Electronics Letters*, Vol. 41, No.18, pp. 1016-1017, 2005

[6] S. Schwartz and Y. S. Yeh, "On the distribution function and moments of power sums with log-normal components," *Bell Syst. Tech. J.*, Vol. 61, pp. 1441-1462, 1982.