GENERALIZED HELICAL MODEL OF THE YARN STRUCTURE

Zdeněk KALOUSEK[1]
[1] *Technical University of Liberec, Faculty of Mechanical Engineering, Department of Applied Cybernetics, Liberec, Czech Republic*
e-mail zdenek.kalousek@tul.cz

Abstract:
The geometry of fibers inside the yarn is commonly described by the idealized helical model. This model does not admit any migration of the fibers and it cannot explain the coherence of the staple yarns. The proposed model is based on the helices too, but the helices may have various axes (parallel to the yarn axis). If the distribution of the helices parameters (radius, axis position and thread) is known, the structural models of the mechanic properties of the yarn may be created, even for the staple yarns.
Key words: yarn, model, helix, probability density function

1. Idealized helical model

The modelling of various properties ot the yarns needs some description of the internal structure of the yarn. The simplest commonly used model is the idealized helical model. It supposes the fibres in the yarn to be positioned on the helices with common axis identical to the yarn axis.

In this model, all the fiber's helices with the radius a must be characterized by the same angle $\beta(a)$ between the fiber helix and the yarn axis. The dependece $\beta(a)$ can be arbitrary, but for the yarns especially, the constant reduced height b of the helix is usually assumed:

$$\beta = \operatorname{atan}\left(\tfrac{a}{b}\right) . \tag{1}$$

Generally, the reduced length of the helix may be some function of the radius a: $b = b(a)$, or – if we use the angle β as the more natural quantity for the description of the fibre in the yarn: $b = a \operatorname{cotan}(\beta(a))$ (see the Fig. 1).

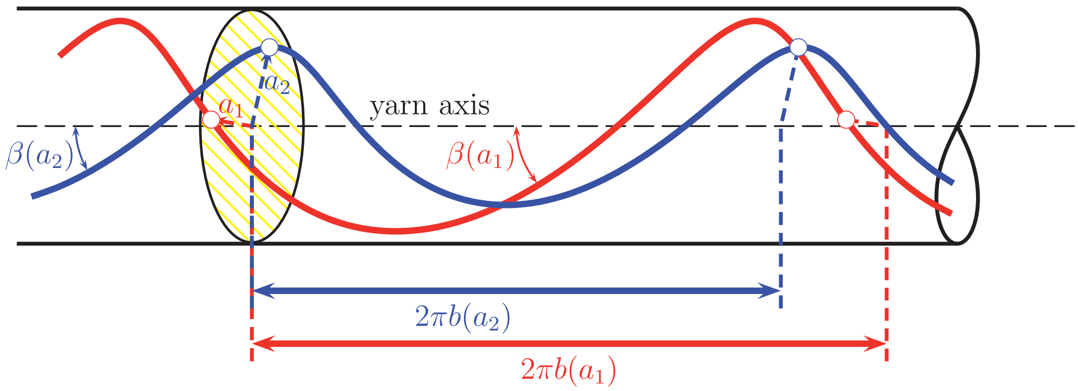

Figure 1: Individual fibers in the yarn – idealized helical model

This model is adequate for some twisted structures, e.g. for the ropes produced from the infinitely long fibers[1], but it cannot describe the staple yarns adn their properties. The

[1] all the fibers connect the ends of the monitored segment

most important is the fact, that it does not allow the radial migration of the fibres. The radial migration is really observed, but it is not the only one reason for its importance. The radial migration of the fibers ensures the elasticity and consistence of the staple yarns.

Let us imagine the elongation of the yarn. When this elongation is not strictly plastic, then its elastic part is caused by the elasticity of fibers. Therefore the fibers must be elongated too, and the elongation is a product of some force applied to the fiber. When the staple yarn is elongated, the only one force acting on the fibers is the friction.[2]

The friction is result of the action of forces oriented perpendicularly to the fiber axis – these forces may be decomposed to the tangential part (between the fibers with the same distance from the yarn axis) and the radial one (between the neighbouring fibers with different distances from the yarn axis). The tangential forces are usually smal, the result [1] shows, that especially in the outer part of the twisted stucture the tangential contact between the fibres disappears completely. The radial forces have character of the radial presure caused by strained bent fibers. But there does not exist any radial pressure acting at the fibers on the yarn surface, therefore these fibers cannot be strained, they cannot press their closest neighbours, . . . and the whole structure desintegrates.

The conclusion of this speculation is clear – the radial migration of the fibers in the staple yarns is necessary.

2. Fully generalized helical model

Despite of the weakness mentioned in the preceeding section, the idealized helical model has one advantage – it can be realized practically. All the more complicated models ought to be constructed with respect to the fact, that each point of the space can belong to only one fiber. The creation of some set of more general (and simply described) fiber's curves satisfying the mentioned condition is practically impossible.

The proposed idea is the creation of the continuous set of the possible fiber's curves. Each curve will be characterized by its probability density function expressing the probability of location of some fiber's segment on the curve. Then different curves may intersect one another, but the total probability obtained by the appropriate integration of the probability densities for all the curves passing by each point must not exceed 1. This model can describe effectively the yarn filling too.

The simplest set of the fiber's curves is the set of helices with their axes oriented paralelly to the yarn axis and positioned in various distances from the yarn axis, with different radii, and with various slope angles. The fully generalized model can suppose variable probability density function in different parts of one helix – for example the location of the fiber segment on the helix may be more probable in smaller distance from the yarn axis, and for the bigger distances is more probable some another helix. The fiber can change its assigned helices continuously.

Each smooth curve may be composed from the segments of the mentioned helices (Fig. 2). In any point of the fiber curve the assigned helix is defined by the tangent identical to the fiber's tangent and the floor projection of the helix is the osculating circle of the fiber's curve floor projection. The identity of both the tangents is important for the modelling of the mechanical behaviour of the yarns. The curvature of general curve neednot have its radiusvector perpendicular to the yarn axis (it is typical for the created helical model), but this component of the curvature is the most important for the fibers in the yarns.

Each helix from the system is defined by its slope angle β, radius a and distance of its

[2] except the ends of the elongated yarn segment

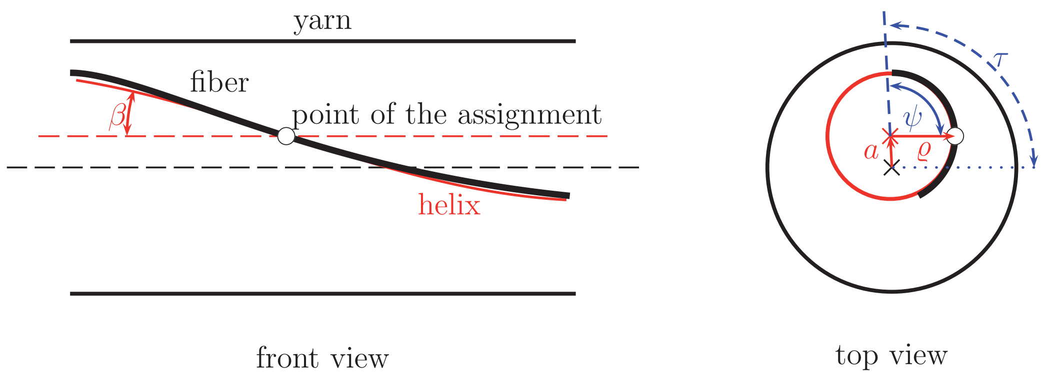

Figure 2: Assignment of the helix to the general curve – generalized helical model

axis from the yarn axis ϱ (Fig. 2). If the different parts of the helix are not equivalent, as mentioned above, the segment (or better the point) of helix is characterized by the angle ψ between the vector connecting the floor projections of the helix and the yarn axes and the vector connecting the floor projections of the selected point and the helix axis.

The definition of the helix is completed by the description of the position of its axis in the yarn body related to some typical direction in the space. This position may be defined by the angle τ (Fig. 2), but in the standard yarn, all the direction are equivalent, therefore this quantity is only auxiliary.

We can define the probability density function

$$\widetilde{f}(\beta, a, \varrho, \psi, \tau) \tag{2}$$

now. The probability of the assignment of a helix described by the group of five parameters $[\beta, a, \varrho, \psi, \tau]$ belonging the set $\widetilde{\Omega} \subset \mathbf{R}^5$ to the fiber segment is then

$$P = \iiiint\!\!\!\int_{\widetilde{\Omega}} \widetilde{f}(\beta, a, \varrho, \psi, \tau) \mathrm{d}\widetilde{\Omega} \ . \tag{3}$$

If we assume all the direction τ in the yarn to be equivalent, the function $\widetilde{f}(\beta, a, \varrho, \psi, \tau)$ does not depend on the parameter $\tau \in \langle 0, 2\pi \rangle$, and the probability density

$$f(\beta, a, \varrho, \psi) = 2\pi \widetilde{f}(\beta, a, \varrho, \psi, \tau) \tag{4}$$

is more suitable; the probability of the assignment of a helix described by the tetrad of parameters $[\beta, a, \varrho, \psi]$ belonging the set $\Omega \subset \mathbf{R}^4$ to the fiber segment is then expressed analogously to (3)

3. Determination of the probability density

There exist various methods for the description of the fiber's curves or their segments in the yarn; the most common are the use of marked fibers ([2] and others papers), and the sequence of the normal cross-sections of the yarn, e. g. [3]. The measurment can give a sufficiently large set of data for the segments with well defined values $[\beta, a, \varrho, \psi]$. Then the statistical methods for the calculation of the probability density function are available.

Nevertheless, the mentioned methods have their disadvantages. The methods of the type [2] need the especially prepared yarns and they are not applicable to the usual commercial yarns. The sequences of the normal cross-sections are applicable for any yarn, but they are very time-consuming.

The method of the oblique cross-sections [4], [5] is universally applicable and significantly easier than the method [3], but it does not give the curvatures; only the orientation of the fiber's curve tangent in given point is expressed. By the appropriated statistical methods we can get the probability density function $\breve{f}(\beta, r, \varphi)$, where r denotes the distance of the selected point from the yarn axis and φ is the angle between the radiusvector of the observed point and the floor projection of the fiber's curve tangent (see Fig. 3).

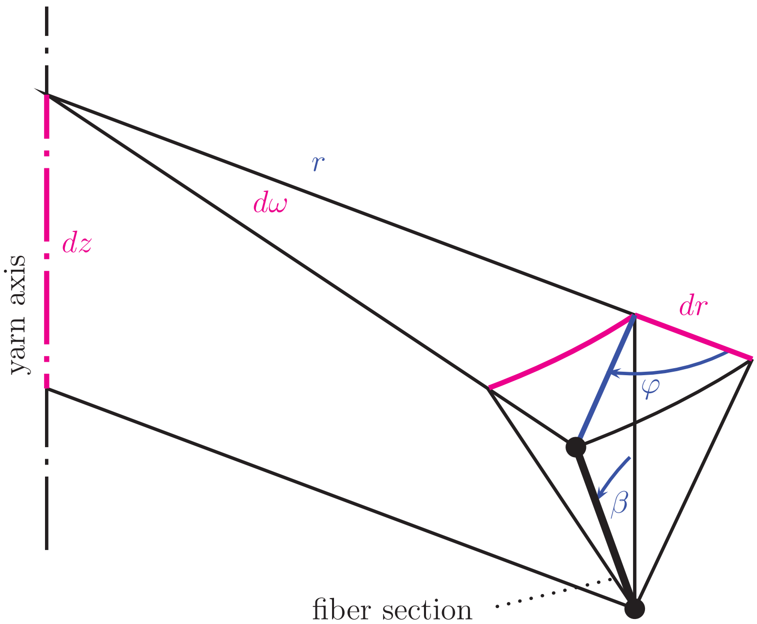

Figure 3: Parameters measured by the oblique cross-sections method

We add the radius ϱ of the helix to the parameters measured by the oblique cross-sections method temporarily (see Fig. 4). The tetrads $[\beta, a, \varrho, \psi]$ and $[\beta, r, \varrho, \varphi]$ describe the helix equivalently. We can imagine now, that the probability density function $\underline{f}(\beta, r, \varrho, \varphi)$ is known. In this case the relation between the probability densities $\underline{f}(\beta, r, \varrho, \varphi)$ and $f(\beta, a, \varrho, \psi)$ exists:

$$\underline{f}(\beta, r, \varrho, \varphi) = f(\beta, a, \varrho, \psi) \left| \frac{D(\beta, a, \varrho, \psi)}{D(\beta, r, \varrho, \varphi)} \right| , \tag{5}$$

where the factor $\left| \frac{D(\beta,a,\varrho,\psi)}{D(\beta,r,\varrho,\varphi)} \right|$ is the Jacobian of the transformation $[\beta, a, \varrho, \psi] \to [\beta, r, \varrho, \varphi]$. This transformation is expressed by

$$a = \sqrt{r^2 + \varrho^2 - 2r\varrho \sin\varphi} , \tag{6}$$

$$\psi = \arcsin\left(\frac{r}{a}\cos\varphi\right) = \arcsin\left(\frac{r\cos\varphi}{\sqrt{r^2 + \varrho^2 - 2r\varrho\sin\varphi}}\right) . \tag{7}$$

The parameters β, ϱ remain unchanged, therefore the Jacobian is

$$\left| \frac{D(\beta, a, \varrho, \psi)}{D(\beta, r, \varrho, \varphi)} \right| = \left| \begin{matrix} \frac{\partial a}{\partial r} & \frac{\partial a}{\partial \varphi} \\ \frac{\partial \psi}{\partial r} & \frac{\partial \psi}{\partial \varphi} \end{matrix} \right| =$$

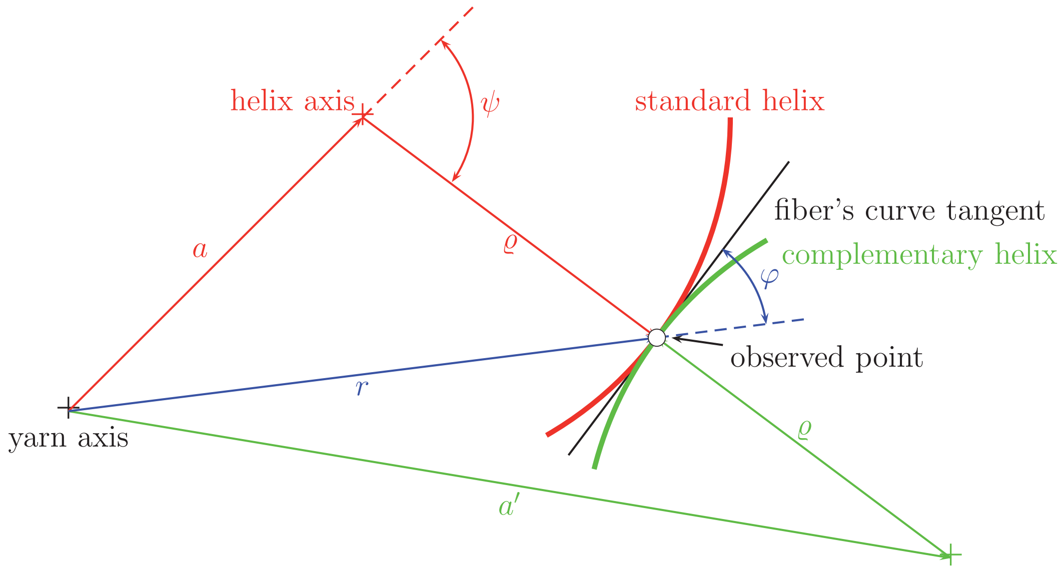

Figure 4: Oblique cross-sections method parameters and the generalized helical model prameters – top view

$$= \begin{vmatrix} \frac{r-\varrho\sin\varphi}{\sqrt{r^2+\varrho^2-2r\varrho\sin\varphi}} & -\frac{r\varrho\cos\varphi}{\sqrt{r^2+\varrho^2-2r\varrho\sin\varphi}} \\ \frac{\varrho\cos\varphi(\varrho-r\sin\varphi)}{|r\sin\varphi-\varrho|(r^2+\varrho^2-2r\varrho\sin\varphi)} & \frac{-r\sin\varphi\left(r^2+\varrho^2-2r\varrho\sin\varphi\right)+r^2\varrho\cos^2\varphi}{|r\sin\varphi-\varrho|(r^2+\varrho^2-2r\varrho\sin\varphi)} \end{vmatrix} =$$
$$= \frac{r}{\sqrt{r^2+\varrho^2-2r\varrho\sin\varphi}} \,. \tag{8}$$

The use of the method which does not give the information about the helix radius ϱ acquires the adequate integration over all the possible radii ϱ. We must remind that in the point described by the parameters r and φ two different helices with the same radius ϱ exist – it is the "standard" helix displayed in the Fig. 4 in red, and the "complementary" helix displayed in green. The complementary helix is characterized by the opposite slope $\beta' = -\beta$. Then the mentioned integration gives the probability density measured by the oblique cross-sections method

$$\breve{f}(\beta,r,\varphi) = \int\limits_{\text{standard}} \underline{f}(\beta,r,\varrho,\varphi)\mathrm{d}\varrho + \int\limits_{\text{complement}} \underline{f}(-\beta,r,\varrho,\varphi)\mathrm{d}\varrho \,, \tag{9}$$

$$\frac{\breve{f}(\beta,r,\varphi)}{r} = \int\limits_0^\infty \frac{f(\beta,\sqrt{r^2+\varrho^2-2r\varrho\sin\varphi},\varrho,\psi)}{\sqrt{r^2+\varrho^2-2r\varrho\sin\varphi}}\mathrm{d}\varrho + $$
$$+ \int\limits_0^\infty \frac{f(-\beta,\sqrt{r^2+\varrho^2+2r\varrho\sin\varphi},\varrho,\psi)}{\sqrt{r^2+\varrho^2+2r\varrho\sin\varphi}}\mathrm{d}\varrho \,; \tag{10}$$

the last formula is based on (5) and (8)

If we want to express the unknown probability density function $f(\beta,a,\varrho,\psi)$ from the measured values $\breve{f}(\beta,r,\varphi)$, the integral equation (10) does not give enough information – the unknown function has one parameter more. Nevertheless, we can assume the probability density function to be independent on the angle ψ,[3] and define the probability

[3]This assumption is advantageous, because it makes possible to avoid the solution of the "conservation of the fibers number" problem existing by the use of general probability density $f(\beta,a,\varrho,\psi)$.

density function

$$\hat{f}(\beta, a, \varrho) = \int_0^{2\pi} f(\beta, a, \varrho, \psi)\mathrm{d}\psi = 2\pi f(\beta, a, \varrho, \psi) \; . \tag{11}$$

Then the equation (10) is modified to

$$\begin{aligned} \frac{2\pi \breve{f}(\beta, r, \varphi)}{r} &= \int_0^{\infty} \frac{\hat{f}(\beta, \sqrt{r^2 + \varrho^2 - 2r\varrho\sin\varphi}, \varrho)}{\sqrt{r^2 + \varrho^2 - 2r\varrho\sin\varphi}} \, \mathrm{d}\varrho + \\ &+ \int_0^{\infty} \frac{\hat{f}(-\beta, \sqrt{r^2 + \varrho^2 + 2r\varrho\sin\varphi}, \varrho)}{\sqrt{r^2 + \varrho^2 + 2r\varrho\sin\varphi}} \, \mathrm{d}\varrho \; . \end{aligned} \tag{12}$$

The (not fully) generalized helical model of the yarn structure will be characterized by the probability density function $\hat{f}(\beta, a, \varrho)$ now.

4. Calculation of the probability density for the generalized helical model

First, it is needed to say that there does not exist any direct formula for the expression of the probability density function $\hat{f}(\beta, a, \varrho)$, when the probability density $\breve{f}(\beta, r, \varphi)$ is known.

Before the start of any calculation, we must note that the expression on the right-hand side of (12) depends on the variable φ by means of the function *sin*, i. e. the probability density of the left-hand side of (12) must satisfy the condition

$$\breve{f}(\beta, r, \varphi) = \breve{f}(\beta, r, \pi - \varphi) \; . \tag{13}$$

If this condition is not satisfied, the introduced generalized helical model cannot be used, the fully generalized helical model is necessary and the data obtained from the oblique cross-sections measurement cannot be sufficient for the calculations.

For to avoid the difficulties caused by the possible zero value of the denominators of the fractions on the right-hand side of (12), we can define the auxiliary function $\widehat{F}(\beta, a, \varrho)$ so that

$$\hat{f}(\beta, a, \varrho) = a\widehat{F}(\beta, a, \varrho) \; , \tag{14}$$

then

$$\begin{aligned} \frac{2\pi \breve{f}(\beta, r, \varphi)}{r} &= \int_0^{\infty} \widehat{F}(\beta, \sqrt{r^2 + \varrho^2 - 2r\varrho\sin\varphi}, \varrho) \, \mathrm{d}\varrho + \\ &+ \int_0^{\infty} \widehat{F}(-\beta, \sqrt{r^2 + \varrho^2 + 2r\varrho\sin\varphi}, \varrho) \, \mathrm{d}\varrho \; . \end{aligned} \tag{15}$$

As mentioned above, the problem of finding the function $\hat{F}(\beta, a, \varrho)$ is solvable only by the numeric way. The possible approximative solution is described in the following text.

The suggested method is based on the Riesz' theorem and the suitable base of the space of the admissible probability density functions similarly to the finite elements method. Let $\mathcal{E} = \{e_i\}_{i=1}^N$ be an independent set of limited functions $e_i(\beta, a, \varrho)$ non-negative on $\mathbf{R}^3$ with bounded support. The set $\mathcal{E}$ defines the set $\mathcal{G}$ of the limited non-negative functions with bounded support

$$g_i(\beta, r, \varphi) = \frac{r}{2\pi} \int_0^{\infty} \left[e_i(\beta, \sqrt{r^2 + \varrho^2 - 2r\varrho\sin\varphi}, \varrho) + e_i(-\beta, \sqrt{r^2 + \varrho^2 + 2r\varrho\sin\varphi}, \varrho) \right] \mathrm{d}\varrho \; . \tag{16}$$

When the functions g_i are independent, they generate a basis of the space of the functions $\breve{f}(\beta, r, \varphi)$ (in the opposite case we can exclude the redundant functions e_i and g_i).

The oblique cross-sections measurement gives the isolated triplets $[\beta_k, r_k, \varphi_k]$ with the corersponding weights w_k; the probability density $\breve{f}(\beta, r, \varphi)$ is calculated from this set consequently. Nevertheless, the set of the measurement results may be treated by another way – we can consider it as the sum of the Dirac's distributions

$$\bar{f}(\beta, r, \varphi) = \frac{\sum_{k=1}^{M} w_k \delta(\beta - \beta_k)\delta(r - r_k)\delta(\varphi - \varphi_k)}{\sum_{k=1}^{M} w_k} ; \tag{17}$$

M represents the number of measured fiber's segments.

The expression $\bar{f}(\beta, r, \varphi)$ defined by (17) is a functional acting on the space of continuous functions of three variables; for the continuous function $u(\beta, r, \varphi)$ the value of the functional $\bar{f}(u)$ is defined as

$$\bar{f}(u) = \frac{\sum_{k=1}^{M} w_k u(\beta_k, r_k, \varphi_k)}{\sum_{k=1}^{M} w_k} . \tag{18}$$

Let us reduce the admissible functions u to the space $\mathcal{U}$ generated by the set $\mathcal{G}$. The functions g_i generating the space $\mathcal{U}$ are integrable in quadrate, therefore the scalar product may be defined on $\mathcal{U}$ and the Riesz' theorem is valid: There exists a function $\bar{u}_f \in \mathcal{U}$ so that

$$\bar{f}(u) = \langle \bar{u}_f, u \rangle \tag{19}$$

for any $u \in \mathcal{U}$. The function $\bar{u}_f$ is a linear combination of the functions g_j

$$\bar{u}_f = \sum_{j=1}^{N} c_j g_j(\beta, r, \varphi) \tag{20}$$

and if the equation (19) is valid for any function $u \in \mathcal{U}$, we will consider especially all the functions $g_i(\beta, r, \varphi)$. Then we get from (18) and (19)

$$\frac{\sum_{k=1}^{M} w_k g_i(\beta_k, r_k, \varphi_k)}{\sum_{k=1}^{M} w_k} = \iiint_{\mathbf{R}^3} \sum_{j=1}^{N} c_j g_j(\beta, r, \varphi) g_i(\beta, r, \varphi) \, \mathrm{d}\beta \, \mathrm{d}r \, \mathrm{d}\varphi \tag{21}$$

for any $i = 1, 2, \ldots, N$. The relation (21) represents a system of linear equations

$$\sum_{k=1}^{M} w_k \sum_{j=1}^{N} a_{ij} c_j = \sum_{k=1}^{M} w_k \cdot g_i(\beta_k, r_k, \varphi_k) \tag{22}$$

with the positive symmetric matrix A with the elements

$$a_{ij} = \iiint_{\mathbf{R}^3} g_i(\beta, r, \varphi) g_j(\beta, r, \varphi) \, \mathrm{d}\beta \, \mathrm{d}r \, \mathrm{d}\varphi . \tag{23}$$

The function $\bar{u}_f(\beta, r, \varphi)$ defined by the Riesz' theorem may be considered as the "better achievable approximation" of the probability density function obtained from the oblique cross-sections measurement. Its expression by (20) gives directly the corresponding probability density function

$$\hat{f}(\beta, a, \varrho) = a \sum_{j=1}^{N} c_j e_j(\beta, a, \varrho) \ . \tag{24}$$

5. Remaining problems to solve

The presented process allows to create the relatively general model of the fibers geometry inside the yarn involving the radial fibers migration. This model may be simply used for the following calculations of the mechanical properties of the yarns. The given guideline for the probability density function calculation generates however some unsolved questions:

- The probability density is a positive function, then the coefficients c_j must belong to a specific set. If the set $\mathcal{E}$ is constructed from the standard finite-element functions, then $c_j \geq 0$ for any $j = 1, 2, \ldots, N$. If the obtained solution of the system (22) does not satisfy this condition, the (uniquely existing) function $\bar{\bar{u}}_f$ closest to the calculated $\bar{u}_f$ and characterized by al the coefficients $c_j \geq 0$ must be found.
- The dimension N of the used space $\mathcal{E}$ is not yet characterized. If the dimension is too small, the obtained probability density is not very apposite; if this dimension is too large, the individual measurements are accentuated and the danger of the negative values c_j increases.

Despite of the mentioned problems, the suggested model represents a powerful mean of the modelling of the structure-depending properties of the yarns.

6. **References:**

[1] Kalousek, Z., Vyšanská, M: *The Ropes Bear More*, Conference "Engineering Mechanics", Svratka, 2006.

[2] Morton, W. E., Summers, R. J.: *J.Text. Inst.*, 40 (1949), p. 106.

[3] Pavlov, N. T., Pejsachov, V. K.: *Textile Industry*, 9–10 (1976), p. 17 (russian).

[4] Vyšanská, M.: *Behaviour of Fibres in Yarn through the Oblique Cross-sections*, 4th conference "AUTEX", Roubaix, 2004.

[5] Kalousek, Z.: *Information Contained in the Oblique Cross-section of Yarns*, 4th conference "AUTEX", Roubaix, 2004.