Effective properties of multilayered composites in spherical coordinates

Alejandro Alonso Fúster, David Guinovart Sanjuán, Raúl Guinovart Díaz,
Julián Bravo Castillero, Reinaldo Rodríguez Ramos
Facultad de Matemática y Computación, Universidad de La Habana. San Lázaro y L, Vedado, Habana 4. CP 10400. Cuba

Introduction

The problem for determining the effective properties of a shell composite material with N concentric layers and thickness δ is studied. The properties change according to radial position. We consider that C depends only on the component $x^3 = r$, where r is the radial coordinate, has the periodicity property

$$\mathrm{C}(r+n\delta)=\mathrm{C}(r) \tag{1}$$

where $n = 1,2,\dots,N$ and $R_n = R_0 + n\delta$, R_0 is the radius of the first inner layer with the condition $R_0 < \cdots R_n < \cdots R_N$. A composite that satisfies the condition (1) is called quasi-periodic structures. The periodic cell Y represents inner concentric rings of radius R_n and thickness δ for each n –layers; its area S_n augment according to the relation

$$S_n = \pi\delta\left(R_0+\delta+2n\delta\right). \tag{2}$$

Let us now analyze the shell composite and for that purpose we introduce a curvilinear coordinate system $\vec{x} = (x^1, x^2, x^3) \in \Omega$. We introduce the fast curvilinear coordinate system $\vec{\xi} = (\xi^1, \xi^2, \xi^3) \in \mathrm{Y}$,

$$\xi^\beta = \frac{x^\beta}{\alpha} + n_\beta \frac{a^\beta}{\alpha},\ (\beta = 1,2,3) \tag{3}$$

where α is a small parameter defined as $\alpha = l/\mathrm{L}$; l and L are the cell and composite lengths, respectively; $\vec{a} = (a^1, a^2, a^3)$ are the dimensions of the quasi-periodic cell and we consider l as $\mathrm{Max}\,\vec{a}$ [1].

Consider a boundary value problem of nonlinear elasticity theory in curvilinear coordinates system given in the following form [1]:

$$\sigma^{ij}_{,j} + 2\Gamma^{(i}_{jp}\sigma^{j)p} + F^i = 0, \tag{4}$$

with boundary conditions

$$u_i\,|_{\Sigma_1} = u_i^0, \mathrm{C}^{ijmn}\left(u_{m,n} - \Gamma^p_{mn}u_p\right)\mathrm{n}_j\,|_{\Sigma_2} = \mathrm{S}_0^i, \tag{5}$$

where the elastic modulus tensor $\mathrm{C} = \mathrm{C}(\vec{\xi}, \vec{x})$ and the solution $u(\vec{x})$ is defined in an open and bounded region $\Omega \in \mathbb{R}^3$, with smooth surface $\Sigma_1 \cup \Sigma_2$; Γ^p_{mn} are the Christoffel´s symbols. S_0^i is a prescribed force on the surface.

Using the asymptotic homogenization method (AHM) the solution of the problem (4) is searched. In this sense, a two scales asymptotic expansion up to order 2 is proposed in the form [2]:

$$u_m(\vec{x}) = \upsilon_m(\vec{x}) + \varepsilon\left[\mathrm{N}^l_{(0)m}(\vec{\xi},\vec{x})\upsilon_l(\vec{x}) + \right.$$
$$\left. + \mathrm{N}^{lk}_{(1)m}(\vec{\xi},\vec{x})\upsilon_{l,k}(\vec{x})\right] + o(\varepsilon). \tag{6}$$

Where $v_m(\vec{x})$, $\mathrm{N}^l_{(0)m}(\vec{\xi},\vec{x})$ and $\mathrm{N}^{lk}_{(1)m}(\vec{\xi},\vec{x})$ are sufficiently smooth functions and satisfy [2]:

$$\left\langle \mathrm{N}^l_{(0)m}(\vec{\xi},\vec{x})\right\rangle = \left\langle \mathrm{N}^{lk}_{(1)m}(\vec{\xi},\vec{x})\right\rangle = 0, \tag{7}$$

The angular bracket $\langle f(\vec{\xi},\vec{x})\rangle = \frac{1}{\mathrm{V}}\int f(\vec{\xi},\vec{x})d\mathrm{V}$ denotes the average of the function in the periodic cell Y over the volume Ω.

The AHM is used in order to obtain the analytic expression of the effective coefficients for a shell composite, wish are employed to obtain the homogenized counterplans of problem (4) (see eq. (6.45) page 133 of [1]). Two sets of effective coefficients are obtained. We denote by H^{ijkl}, the effective properties which are the same to the problem of a laminated composite in Cartesian coordinates system [1] where the direction $x^3 = r$ perpendicular in each point to the distribution of the laminates is considered

$$\mathrm{H}^{ijkl} = \left\langle \mathrm{C}^{ijkl}\right\rangle - \left\langle \mathrm{C}^{ijm3}\left(\mathrm{C}^{m3r3}\right)^{-1}\mathrm{C}^{r3kl}\right\rangle +$$
$$+ \left\langle \mathrm{C}^{ijm3}\left(\mathrm{C}^{m3n3}\right)^{-1}\right\rangle \left\langle \left(\mathrm{C}^{n3p3}\right)^{-1}\right\rangle^{-1} \cdot \tag{8}$$
$$\cdot \left\langle \left(\mathrm{C}^{p3q3}\right)^{-1}\mathrm{C}^{q3kl}\right\rangle$$

Moreover, the second set of effective coefficients which contains the geometric information of the composite is obtained by the following relation

$$\mathrm{H}^{ijk}_*(\vec{x}) = -\Gamma^k_{pq}\,\mathrm{H}^{ijpq}. \tag{9}$$

Therefore, the effective coefficients H^{ijk}_* are associated to the shell composite in curvilinear coordinates.

Composite shell with spherical shape

Now, the spherical coordinates system is introduced in the form (θ, ϕ, r). The metric tensor is a square diagonal matrix of order 3 with nonzero components $g_{11} = r^2 sen^2\phi$, $g_{22} = r$, $g_{11} = 1$.

The corresponding Christoffel´s symbols are:

$$\Gamma^3_{22} = -r, \Gamma^3_{11} = -r\,\text{sen}^2\,\phi, \Gamma^3_{22} = -\text{sen}\,\phi\cos\phi,$$

$$\Gamma^2_{23} = \Gamma^1_{13} = \frac{1}{r}, \quad \Gamma^1_{12} = \cot\phi. \qquad (10)$$

Now, we consider the constituents of the composite isotropic and we have a quasi-periodic medium composed by n layers of thickness δ with different properties for each layer (in this case, E and ν denote the Young's modulus and Poisson ratio, respectively). The list of all effective elastic components H^{ijkl} is given in [1]. The following material parameters are used for the numerical computations of the effective properties in a shell composite with three isotropic homogeneous layers:

Parameters	Values
Young Modulus E (N/mm^2) [5]	0.275, 0.622, 0.279
Thickness (mm) [4,5]	495, 627
Poisson ratio ν [3, 4, 5]	0.49
Initial radio R_0 (mm) [3]	7.6125, 7.5465

The parameters showed on previous table respond to human cornea's studies. Considering three layers: Epithelium-Bowman's membrane, Stroma, Endothelium-Descemet's membrane with 10%, 87% and 3% of the full corneal thickness, respectively; where the Stroma is the layer more strength and therefore it is related to Young's modulus of one order of magnitude bigger than the remaining layers and all the layers of the cornea were assigned with the same Poisson's ratio to indicate near incompressibility [4]. Initial radio R_0 has been considered as $R_0 = R_{\text{curvature}} - \text{Thickness}/2$.

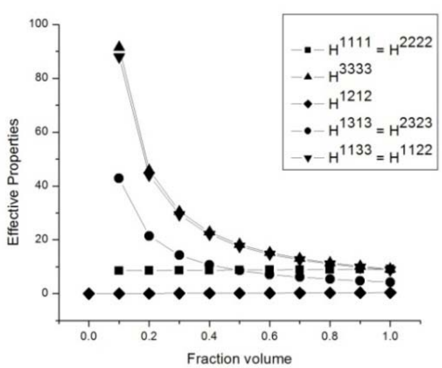

Fig.1. Effective coefficients $\mathbf{H}^{ijkl}$ associated to Cartesian coordinates laminated composite.

Fig. 1 shows the behavior of the effective coefficients H^{ijkl} calculated by formulae (8) taking the fraction volume as:

$$\text{V} = \frac{\phi\delta\left(R_0 + \delta + 2n\delta\right)}{\text{S}_n}, \quad 0 \le \phi \le \pi/2. \qquad (11)$$

The relation (9) allows us to obtain effective coefficients H^{ijk}_{*} associated to curvilinear coordinates (in particular spherical coordinates) related to the effective coefficients H^{ijkl}. An analysis of the behavior of the effective properties given by H^{ijk}_{*} through the shell in the direction of the angle ϕ $\left(0 \le \phi \le \frac{\pi}{2}\right)$, i.e. from the apex to the external border of the shell is displayed in Fig.2.

$$\text{H}^{112}_{*} = -\Gamma^2_{11}\,\text{H}^{1111} = \text{sen}\,\phi\cos\phi\cdot\text{H}^{1111},$$

$$\text{H}^{113}_{*} = -\Gamma^3_{11}\,\text{H}^{1111} = r\,\text{sen}^2\,\phi\cdot\text{H}^{1111}, \qquad (12)$$

$$\text{H}^{121}_{*} = -\Gamma^1_{12}\,\text{H}^{1212} = \cot\phi\,\text{H}^{1212}.$$

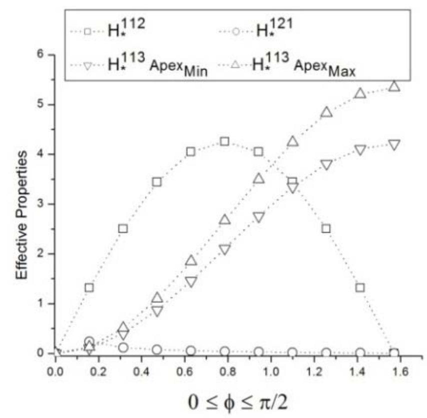

Fig.2. Effective coefficients $\mathbf{H}^{ijk}_{*}$ associated to spherical coordinate case vs. ϕ angle.

Conclusions

The effective coefficients for shell composite are calculated using the asymptotic homogenization approach. The contribution of the geometric nonlinearity for the elastic quasi periodic laminated composite is reflected in the overall properties. The curvature of the composite affects considerable the global behavior of the elastic properties. An application of the present model to the study of the human cornea is considered.

References

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