MATRIX ANALYSIS OF STRUCTURAL COMPOSITE LAMINATES

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Abstract

A compact method based on Classical Laminate Theory (CLT) is developed here to directly relate the external loads to strains and stresses at any surface location of a flat structural composite laminate. Besides in-plane forces and moments applied along the laminate edges, loads induced by changes in temperature and moisture contents are included in the load vector, consisting of eight components. Matrices and vectors are employed to write the equations in a compact form.

The final outcomes are stress and strain coefficient matrices, each element of which directly gives the stress or strain for a unit magnitude of a load component.

Keywords

Composite laminate, matrix method of analysis, classical laminate theory, stresses and strains, coefficient matrix.

1. INTRODUCTION

Figures 1 and 2 show the laminate geometry and applied loading (Mallick, 2003). The surface is defined by a parameter $z(j)$, where j is the number of lamina in which the surface is located and z is its distance from the laminate mid-plane. The stresses and strains are computed in two coordinate axes: X-Y or *loading* coordinates and 1-2 or *material principal* coordinates. For example, $\{\epsilon\}_j$ represents the three strain components $(\epsilon_{xx}, \epsilon_{yy}, \gamma_{xy})$ on a surface $z(j)$ in the j-th lamina, whereas $\{\sigma\}_j^{12}$ represents the three stress components $(\sigma_{11}, \sigma_{22}, \tau_{12})$ of the same surface in 1-2 coordinates.

2. DERIVATION

2.1 Formation of A,B,D Matrices

Define $[A]_j = t_j[\bar{Q}]_j,\ [B]_j = t_j\bar{z}[\bar{Q}]_j,$

$$[D]_j = \left(\frac{t_j^3}{12} + t_j\bar{z}_j^2\right)[\bar{Q}]_j \tag{1}$$

where $[\bar{Q}]_j$ is the stiffness matrix of the j-th lamina, $t_j = z_j - z_{j-1}, \bar{z}_j = \frac{Z_j + Z_{j-1}}{2}$

also, $$[\bar{Q}]_j = [T_\sigma]_j^{-1}[Q_0]_j[T_\epsilon]_j \tag{2}$$

where $[Q_0]_j$, $[T_\sigma]_j$ and $[T_\epsilon]_j$ are the stiffness matrix of j-th lamina assuming $\theta = 0$, and stress and strain transformation matrices respectively.

Then $[A] = \sum_{j=1}^N [A]_j, [B] = \sum_{j=1}^N [B]_j,$

$$[D] = \sum_{j=1}^N [D]_j \tag{3}$$

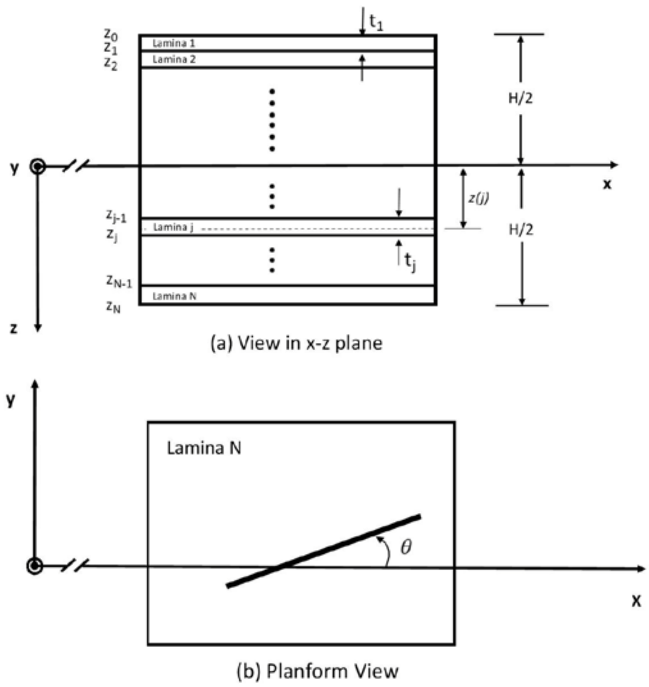

Fig. 1 Laminate nomenclature and definition of surface

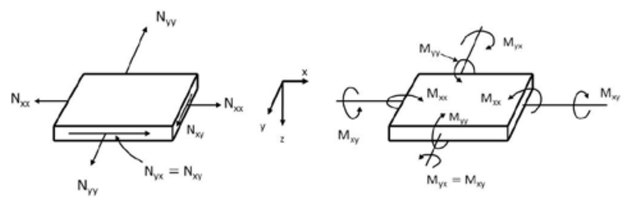

Fig. 2 In-plane, bending and twisting load applied on a laminate

2.2 Vectors of Induced Thermal and Hygral Loads

Next we write down the *unit* thermal and hygral forces and moments as follows.

$$\{\hat{N}^T\} = \sum_{j=1}^N [A]_j\{\alpha\}_j,\ \{\hat{M}^T\} = \sum_{j=1}^N [B]_j\{\alpha\}_j \tag{4a}$$

$$\{\hat{N}^H\} = \sum_{j=1}^N [A]_j\{\beta\}_j,\ \{\hat{M}^H\} = \sum_{j=1}^N [B]_j\{\beta\}_j \tag{4b}$$

where subscripts T and H are for thermal and hygral effects respectively.

Also N = total number of laminas in the stack and $\{\alpha\}_j$ and $\{\beta\}_j$ are the vectors of thermal and hygral coefficients of expansion respectively. Both of these vectors are in X-Y coordinates.

2.3 Derivation of Strains and Stresses

2.3.1 Strains and stresses in X-Y or *loading* coordinates

The stiffness matrix of the laminate is:

$$\left[\frac{A}{B}\middle|\frac{B}{D}\right]\begin{Bmatrix}\epsilon^0 \\ k\end{Bmatrix} = \begin{Bmatrix}N_{\text{total}} \\ M_{\text{total}}\end{Bmatrix} \tag{5}$$

where ϵ^0, k are respectively the *laminate* mid-surface strain vector and curvature vector.

By inverting the matrix equations above

$$\begin{Bmatrix} \epsilon^0 \\ k \end{Bmatrix} = \begin{bmatrix} \frac{A}{B} \middle| \frac{B}{D} \end{bmatrix}^{-1} \begin{Bmatrix} N_{\text{total}} \\ M_{\text{total}} \end{Bmatrix} = \begin{bmatrix} \frac{A_1}{B_1^T} \middle| \frac{B_1}{D_1} \end{bmatrix} \begin{Bmatrix} N_{\text{total}} \\ M_{\text{total}} \end{Bmatrix} \quad (6)$$

Here $\{N_{\text{total}}\} = \{N\} + \{\widehat{N}^T\}\Delta T + \{\widehat{N}^H\}\Delta M$ (7a)

$$\{M_{\text{total}}\} = \{M\} + \{\widehat{M}^T\}\Delta T + \{\widehat{M}^H\}\Delta M \quad (7b)$$

ΔT, ΔM are respectively the changes in temperature and moisture contents.

From Equation (6) above

$$\{\epsilon^0\} = [A_1]\{N_{\text{total}}\} + [B_1]\{M_{\text{total}}\} \quad (8a)$$

and $\{k\} = [B_1^T]\{N_{\text{total}}\} + [D_1]\{M_{\text{total}}\}$ (8b)

The mechanical strain at any surface location $z(j)$ may then be written as

$$\{\epsilon\}_j = \{\epsilon^0 + z(j)k - \alpha_j \Delta T - \beta_j \Delta M\} \quad (9)$$

After substitution of ϵ^0, k, α_j and β_j – all 3×1 vectors, and simplifying:

$$\{\epsilon\}_j = \big[[n]_j \ [m]_j \ \{t\}_j \ \{h\}_j\big]\{L\} \quad (10)$$

where

$$[n]_j = [A_1 + z(j)B_1^T], [m]_j = [B_1 + z(j)D_1]$$

$$\{t\}_j = [n]_j\{\widehat{N}^T\} + [m]_j\{\widehat{M}^T\} - \{\alpha\}_j$$

$$\{h\}_j = [n]_j\{\widehat{N}^H\} + [m]_j\{\widehat{M}^H\} - \{\beta\}_j$$

Finally, from $\{\sigma\}_j = [\bar{Q}]_j\{\epsilon\}_j$

$$\{\sigma\}_j = \big[[\bar{Q}]_j[n]_j \ [\bar{Q}]_j[m]_j \ [\bar{Q}]_j\{t\}_j \ [\bar{Q}]_j\{h\}_j\big]\{L\} \quad (11)$$

where {L} is the generalised load vector $\left(N_{xx}\ N_{yy}\ N_{xy}\ M_{xx}\ M_{yy}\ M_{xy}\ \Delta T\ \Delta M\right)^T$, consisting of eight components.

2.3.2 Strains and stresses in 1-2 or *material principal* coordinates

The transformation from X-Y to 1-2 coordinates is based on the basic rule:

$$\{\epsilon\}_j^{12} = [T_\epsilon]_j\{\epsilon\}_j \text{ and } \{\sigma\}_j^{12} = [T_\sigma]_j\{\sigma\}_j$$

Therefore, the final expressions are:

$$\{\epsilon\}_j^{12} = \big[[T_\epsilon]_j[n]_j \ [T_\epsilon]_j[m]_j \ [T_\epsilon]_j\{t\}_j \ [T_\epsilon]_j\{h\}_j\big]\{L\} \quad (12)$$

and $\{\sigma\}_j^{12} = \big[[\alpha]_j \ [\beta]_j \ \{\gamma\}_j \ \{\delta\}_j\big]\{L\}$ (13)

where

$$[\alpha]_j = [Q_0]_j[T_\epsilon]_j[n]_j , \quad [\beta]_j = [Q_0]_j[T_\epsilon]_j[m]_j$$

$$\{\gamma\}_j = [Q_0]_j[T_\epsilon]_j\{t\}_j , \quad \{\delta\}_j = [Q_0]_j[T_\epsilon]_j\{h\}_j$$

Note that $[\alpha]_j$ and $[\beta]_j$ in Equation (13) are 3×3 matrices as defined above and not the vectors of thermal and hygral coefficients of expansions used in Equations (4a) and (4b).

From Equations (12) and (13) the strain and stress coefficient matrices in 1-2 coordinates are:

$$\big[[T_\epsilon]_j[n]_j \ [T_\epsilon]_j[m]_j \ [T_\epsilon]_j\{t\}_j \ [T_\epsilon]_j\{h\}_j\big]$$

and $\big[[\alpha]_j \ [\beta]_j \ \{\gamma\}_j \ \{\delta\}_j\big]$ respectively.

For any one given surface these are 3×8 matrices. For each additional surface there will be one extra 3×8 matrix. Therefore, if we choose to compute stresses/strains at *M* surfaces we build up the coefficient matrices containing 3*M* rows and 8 columns. Multiplication of coefficient matrix by the appropriate load vector yields the strains or stresses at designated surfaces.

3. EXAMPLES

3.1 Input Material Properties

The following properties of Graphite-epoxy laminate have been taken from the text (Hyer, 2009).

Elastic properties:

E_{11} = 155×10^3 MPa, E_{22} = 12.1×10^3 MPa,
G_{12} = 4.4×10^3 MPa and ϑ_{12} = 0.248.

Thermal and hygral properties:

α_{11} = –0.018×10^{-6}/°C, α_{22} = 24.3×10^{-6}/°C,
β_{11} = 146.0×10^{-6}/%ΔM, β_{22} = 4770×10^{-6}/%ΔM.

All laminas are of the same constituent materials, the same fibre volume fractions and thickness, t = 0.15 mm.

3.2 An Unsymmetric Laminate: $[\pm 30/0]_T$

The three component strains and stresses in the 1-2 coordinates are computed for the upper surface of lamina 2 (from top) for which *z*(*j*) = –0.075 mm.

For this surface the stress coefficient matrix is:

$$\begin{bmatrix} 5.2284 & 0.1431 & -4.8570 & \vdots & -37.7889 & -15.9591 & 102.4685 & \vdots & 0.4523 & \vdots & 146.4266 \\ -0.3066 & 1.6639 & 0.5177 & \vdots & 2.2832 & -6.3108 & 1.7489 & \vdots & -0.1242 & \vdots & -16.2708 \\ 0.3716 & -0.8982 & 0.2263 & \vdots & -2.0194 & 4.9746 & 0.7804 & \vdots & -0.0756 & \vdots & -19.9626 \end{bmatrix}$$

For any given load combinations the stresses on this surface can be found by simply multiplying this matrix by the corresponding load vector. For example, if N_{xx} = 100 N/mm and ΔT = –150°C, all other load components being zero, the stresses are:

σ_{11} = 455.0 MPa, σ_{22} = 12.03 MPa and τ_{12} = 48.5 MPa.

3.3 A Symmetric Laminate: $[\pm 30/0]_S$

For this laminate we compute the stresses in the lower surface of lamina 2, for which *z*(*j*) = –0.15 mm. Again for N_{xx} = 500 N/mm and ΔT = –150°C the stresses are:

σ_{11} = 377.67 MPa, σ_{22} = –17.5 MPa and τ_{12} = 62.24 MPa.

It may be noted that for symmetric laminate the stresses will be the same for mirror-image surface below the mid-plane for which *z*(*j*) = +0.15 mm. However, this is only true if there is no applied bending moment.

4. DISCUSSIONS AND CONCLUSIONS

The matrix formulation presented here for the analysis of any given flat composite laminate under any loading condition is quite straightforward and may readily be performed on a personal computer. The only software required is the one which can manipulate small order matrices, mostly 3×3. Microsoft Excel may be a good choice.

The formulation was originally intended for the application of linear programming to investigate laminate failure based on maximum stress criterion of failure of laminas. This investigation will be reported in another publication.

Bibliography

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