Maximum Amplitude of Solutions on a Ball for a Generalized Lane–Emden Equation

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Abstract I examine the relationship between the amplitude of solutions to the generalized Lane-Emden Dirichlet problem

$$\nabla \cdot (|x|^{\alpha} |\nabla u|^{p-2} \nabla u) + |x|^{\beta} |u|^{q-1} u = 0,\ u|_{\partial \mathcal{B}^n} = 0,\ q > 1,\ 1 < p \leq 2, \alpha, \beta \geq 0$$

where $B^n = \{x \in \mathbb{R}^n : |x| \leq 1\}$, and the number of existing zeros. I prove that the maximum amplitude of a radial solution occurs at the center of the ball and that the greater the number of oscillations of a solution, the greater the amplitude at the origin. I also establish that the maximum amplitude achieved at the origin determines the existence of solutions to the generalized Lane-Emden equation.

1. Introduction

I examine the radial solutions to the nonlinear ordinary differential equation

$$r^{1-n}(r^{n+\alpha-1} |u_r|^{p-2} u_r)_r + r^{\beta} |u|^{q+1} u = 0 \quad (1)$$

with boundary conditions

$$u(1) = u_r(0) = 0 \quad (2)$$

for $1 < p \leq 2 < n$ and $p < q+1 < p^* = \frac{p(n+\beta)}{\alpha+n-p}$, corresponding to the Dirichlet problem

$$\nabla \cdot (|x|^{\alpha} |\nabla u|^{p-2} \nabla u) + |x|^{\beta} |u|^{q-1} u = 0 \quad (3)$$

$$u|_{\partial \mathcal{B}^n} = 0$$

on the unit ball $B^n = \{x \in R^n : |x| \leq 1\}$. I consider bounded solutions such that $u(0)$ and $u_r(0)$ are both finite, with $\alpha, \beta \geq 0, 1 < p \leq 2, q > 1$. In particular, let $u(0) = \sigma, |\sigma| < \infty, u_r(0) = 0, u(1) = 0, u_r(1) = -\delta, -\delta \in (-\infty, \infty)$. Equation (3) serves as a generalization to the Lane-Emden equation. Extensive research on the Lane-Emden equation was done by a number of researchers as it serves to model interesting phenomena in Astronomy, Physics and Engineering. In this paper I prove the existence of a finite number of zeros of a radial solution to (1) and the dependence of such a number on the parameters α, β, p, n, q. I establish a relationship between the number of zeros and the maximum amplitude at the center of the ball. In particular, I prove that the greater the number of oscillations of a radial solution, the greater the amplitude at the origin and that the maximum amplitude of a radial solution is bounded below uniformly by a constant dependent on the parameters p, n, q, α, β.

2. Preliminary results

Definition 1: Let $\Phi_p(x) = |x|^{p-2} x$, for $x \in R, p > 1$ and denote its inverse by $\Phi'_p(x)$, where $\frac{1}{p} + \frac{1}{p'} = 1$.

Definition 2: Define the mapping $T(u(r)) = \sigma - \int_0^r (\frac{1}{t^{\frac{n+\alpha-1}{p-1}}}) \Phi_{p'}(\int_0^r s^{n+\beta-1} \Phi_{q+1}(u(s)) ds) dt$

Suppose that $0 \leq \rho \leq 1$ and $r = r_\rho(\sigma)$ are such that, for a finite σ, we have

$$u(r_\rho(\sigma)) = \rho\sigma, |u(r)| \geq |\sigma| \rho, \forall 0 \leq r \leq r_\rho(\sigma) \quad (4)$$

$$u(r_0(\sigma)) = 0, r_1(\sigma) = 0 \quad (5)$$

Lemma: Let $r_\rho(\sigma)$ be defined as above. Then for some positive constant $C = C(p, n, q, \alpha, \beta)$, $r_\rho(\sigma)$ satisfies

$$|\sigma|^{\frac{-1}{\theta}} (1-\rho)^{\frac{p-1}{\beta+p-\alpha}} \leq C r_\rho(\sigma) \leq |\sigma|^{\frac{-1}{\theta}} \rho^{-\frac{q}{p+\beta-\alpha}} (1-\rho)^{\frac{p-1}{\beta+p-\alpha}} \quad (6)$$

where $C = (\frac{p-1}{p+\beta-\alpha})^{\frac{p-1}{\beta+p-\alpha}} (\frac{1}{n+\beta})^{\frac{1}{\beta+p-\alpha}}$ and $\theta = \frac{\beta+p-\alpha}{q+1-p}$.

Theorem: For $\beta - \alpha + 1 > 0, 1 < p \leq 2, q > 1$, radial solutions of (1) satisfying the boundary condition (2) approach the origin with zero slope such that $u_r(0) = 0$.

Definition 3: The energy function of solutions is $E(r) = \frac{p-1}{p} r^{\alpha} |u_r|^p + \frac{1}{q+1} r^{\beta} |u|^{q+1}$.

Hence,

$$E(r) = \int_0^r [-(\frac{\alpha}{p} + n - 1) s^{\alpha-1} |u_s|^p + \frac{\beta}{q+1} s^{\beta-1} |u|^{q+1}] ds \quad (7)$$

Lemma: $\frac{d}{dr} E(r, \sigma) \leq 0$ for $\alpha, \beta \geq 0, 1 < p \leq 2, q > 1, u(0) = |\sigma| < \infty, u_r(0) = 0, r \geq 0$.

3. Solutions amplitude

Theorem 1: Let $u(r)$ be a solution of (1),(2), suppose that $0 < r_1 < r_2 \leq 1$.

1. If $u(r_1) = u(r_2) = 0$, then $|u_r(r_1)| \geq |u_r(r_2)|$.
2. If $u_r(r_1) = u_r(r_2) = 0$, then $|u(r_1)| \geq |u(r_2)|$.

   Property (2) leads to the final property,
3. $max_{0\leq r\leq 1} |u(r)| = |u(0)| = \sigma$. Hence, the maximum amplitude of solutions occurs at the center of the ball.

Proof: Consider multiply both sides of (1) by $\frac{1}{r^\beta}u_r$, integrate from r_1 to r_2, for $0 \leq r_1 < r_2 \leq 1$, integrate by parts on the left hand side of the resulting equation gives

$$r^{\alpha-\beta} |u_r(r_2)|^p - r^{\alpha-\beta} |u_r(r_1)|^p - \quad (8)$$

$\int_{r_1}^{r_2} r^{\alpha-\beta} |u_r|^{p-2} u_r u_{rr} dr + \int_{r_1}^{r_2} (n + \beta - 1)\ r^{\alpha-\beta-1} |u_r|^p\, dr = -\int_{u(r_1)}^{u(r_2)} |w|^{q-1}\, w dw$

Expanding (1), multiplying by $r^{-\beta-n+1}\frac{u_r}{p-1}$, substituting the resulting equation in (8) and collecting like terms results in

$$\frac{p-1}{p} r^{\alpha-\beta}[|u_r(r_2)|^p - |u_r(r_1)|^p] + (n+\beta-1+\frac{\alpha-\beta}{p})\int_{r_1}^{r_2} r^{\alpha-\beta-1}|u_r|^p\, dr$$

$$+\frac{1}{q+1}[|u(r_2)|^{q+1} - |u(r_1)|^{q+1}] = 0 \quad (9)$$

where $n-1+\frac{\alpha}{p}+\frac{(p-1)\beta}{p} > 0$ for $\alpha, \beta \geq 0, 1 < p \leq 2, q > 1$.

4. Maximum amplitude and existence

Theorem 2: Let $u(r)$ be a nontrivial solution of (1),(2), then $max_{0\leq r\leq 1} |u(r)| \geq [(n+\beta)(\frac{p+\beta-\alpha}{p-1})^{p-1}]^{\frac{1}{q+1-p}}$. That is the maximum amplitude of any solution is bounded below uniformly by $[(n+\beta)(\frac{p+\beta-\alpha}{p-1})^{p-1}]^{\frac{1}{q+1-p}}$. Consequently by property 3 of theorem 1, if $|u(0)| < [(n+\beta)(\frac{p+\beta-\alpha}{p-1})^{p-1}]^{\frac{1}{q+1-p}}$ then $u(r)$ can not be a solution of (1), (2).

Proof: Using the inequality in (6) for $r_\rho(\sigma)$ and letting $\rho = 0$ we have $|\sigma| \geq$

$$[(\frac{1}{r_0(\sigma)})^{\beta+p-\alpha}(\frac{p+\beta-\alpha}{p-1})^{p-1}(\frac{1}{n+\beta})]^{\frac{1}{q+1-p}} \quad (10)$$

Recall that $u(0) = \sigma, r_0(\sigma) = 1$ if u has no zeros in $[0,1]$ and $r_0(\sigma) \leq 1$ in general, then by property 3 of theorem (1) and equation (10) we have $max_{0\leq r\leq 1} |u(r)|$

$$\geq [(\frac{p+\beta-\alpha}{p-1})^{p-1}(\frac{1}{n+\beta})]^{\frac{1}{q+1-p}} \quad (11)$$

5. Existence of a finite number of zeros

Theorem 3: Let $u(r)$ be a nontrivial bounded solution of (1), (2), then u has only finitely many zeros in $[0,1]$ and the total number of zeros of u in $[0,1]$ is no more than the greatest integer less than or equal to $1+\frac{1}{2}(\frac{p-1}{p})^{\frac{p-1}{p}}(|u(0)|)^{\frac{q+1-p}{p}}$.

proof: Suppose that the number of zeros of u in $[0,1]$ is infinite. Then we can choose a sequence $\{r_k\}$ of such zeros which converges to some $r_0 \in (r_k, r_{k+1})$ satisfying $u_r(\eta_k) = 0$ and $\eta_k \to r_0$ as $k \to \infty$. Therefore $u(r_0) = u_r(r_0) = 0$ for some $r_0 \in [0,1]$ and we have $u(r) =$

$$-\int_{r_0}^{r} \Phi_{p'}(\frac{1}{t^{n+\alpha-1}}\int_{r_0}^{t} s^{n+\beta-1}\Phi_{q+1}(u(s)ds)dt \quad (12)$$

for $r \in [0,1]$. Let $\delta > 0$. Then $u(r) \leq$

$$\frac{p-1}{(p+\beta-\alpha)(n+\beta)^{p'-1}}\delta^{\frac{p+\beta-\alpha}{p-1}} Sup_{\xi\in \mathcal{B}_\delta(r_0)} |u(\xi)|^{\frac{q}{p-1}} \quad (13)$$

for $r \in B_\delta(r_0)$, where $B_\delta(r_0) = (r_0-\delta, r_0+\delta) \cap [0,1]$ and $p' = \frac{p}{p-1}$. Since $p' > 0$ and $q > p-1$, let $\delta > 0$ be sufficiently small, then for a non trivial solution where $\|u\|_{\infty, B_\delta(r_0)} \to \infty$ as $\delta \to 0$ then,

$$\|u\|_{\infty,B_\delta(r_0)}^{1-\frac{q}{p-1}} \leq \frac{p-1}{p+\beta-\alpha}\frac{\delta^{\frac{p+\beta-\alpha}{p-1}}}{(n+\beta)^{p'-1}} \to 0 \quad (14)$$

as $\delta \to 0$. This leads to a contradiction, therefore u had finitely many zeros in $[0,1]$. Further, we obtain

$$m \leq 1 + \frac{1}{2}(\frac{p-1}{p})^{\frac{p-1}{p}}(|u(0)|)^{\frac{q+1-p}{p}} \quad (15)$$

6. Maximum amplitude determined by number of oscillation

Theorem 4: The greater the number of oscillations of a radial solution to (1), (2), the greater the value of the maximum amplitude of solution.

Proof: If the first zero of a solution is at $r = R$, then the inequality (11) reads,

$max_{0\leq r\leq 1} |u(r)| \geq$

$$[\frac{1}{R^{\beta+p-\alpha}}(\frac{p+\beta-\alpha}{p-1})^{p-1}(\frac{1}{n+\beta})]^{\frac{1}{q+1-p}} \quad (16)$$

It is now clear that the closer $r_0(\sigma)$ to $r = 0$ the larger the amplitude of the radial solution at the origin.